# Towards double-logarithmic upper bounds on the chromatic number of triangle-free geometric intersection graphs

Bartosz Walczak
Jagiellonian University in Kraków

# Chromatic number vs clique number

$\chi$ chromatic number of a given graph
$\omega$ clique number (= max. size of a clique) of a given graph

Obvious inequality: $\chi \geqslant \omega$

# Chromatic number vs clique number

$\chi$ chromatic number of a given graph
$\omega$ clique number (= max. size of a clique) of a given graph

Obvious inequality: $\chi \geqslant \omega$

Theorem (Zykov, Tutte, Mycielski…)
There exist triangle-free graphs (= graphs with $\omega = 2$) with arbitrarily large chromatic number.

Theorem (Kim 1995)
There exist triangle-free graphs with chromatic number $\Theta(\sqrt{n/\log n})$.

# Chromatic number vs clique number

$\chi$ chromatic number of a given graph
$\omega$ clique number (= max. size of a clique) of a given graph

Obvious inequality: $\chi \geqslant \omega$

Theorem (Zykov, Tutte, Mycielski…)
There exist triangle-free graphs (= graphs with $\omega = 2$) with arbitrarily large chromatic number.

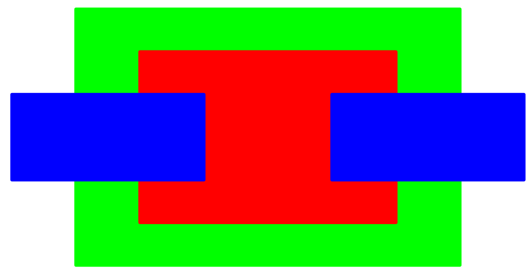
Theorem (Kim 1995)
There exist triangle-free graphs with chromatic number $\Theta(\sqrt{n/\log n})$.

What happens for classes of graphs with geometric representations?

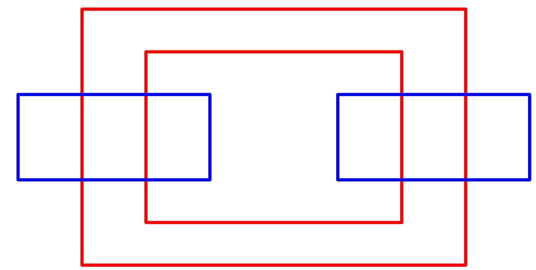
# Geometric intersection graphs

A *geometric intersection graph* has some geometric objects as vertices and all pairs of intersecting objects as edges.

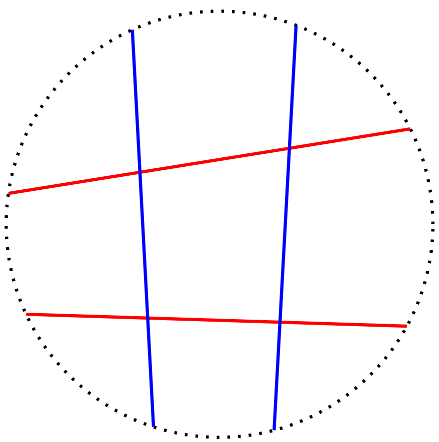
interval graphs



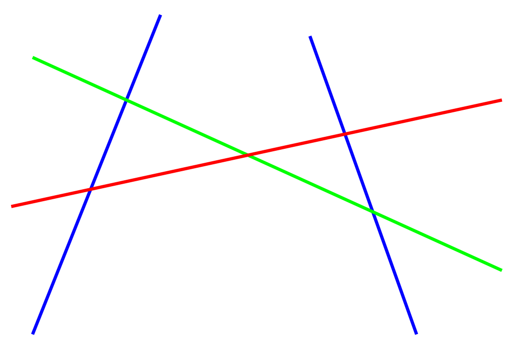
rectangle graphs



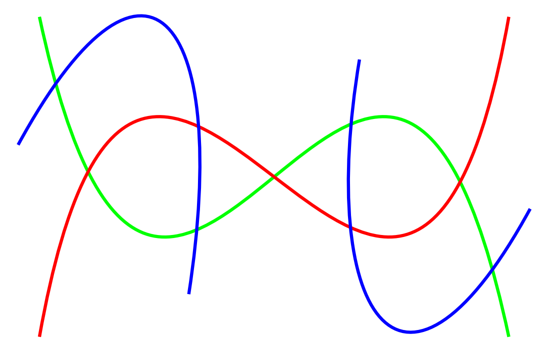
frame graphs



circle graphs



segment graphs



string graphs

# Chromatic number of geometric intersection graphs

Theorem (folklore)
Interval graphs satisfy $\chi = \omega$ (they are perfect).

## Chromatic number of geometric intersection graphs

Theorem (folklore)
Interval graphs satisfy $\chi = \omega$ (they are perfect).

A class of graphs $\mathcal{G}$ is $\chi$-*bounded* if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $\chi \leqslant f(\omega)$ for every graph in $\mathcal{G}$.

## Chromatic number of geometric intersection graphs

Theorem (folklore)
Interval graphs satisfy $\chi = \omega$ (they are perfect).

A class of graphs $\mathcal{G}$ is $\chi$-*bounded* if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $\chi \leqslant f(\omega)$ for every graph in $\mathcal{G}$.

Theorem (Asplund, Grünbaum 1960)
The class of rectangle graphs is $\chi$-bounded.

Theorem (Gyárfás 1985)
The class of circle graphs is $\chi$-bounded.

# Geometric intersection graphs with large chromatic number

Theorem (Burling 1965)
There are triangle-free intersection graphs of boxes in $\mathbb{R}^3$ with chromatic number $\Theta(\log \log n)$.

Theorem (Pawlik et al. 2013)
There are triangle-free intersection graphs of frames, L-figures, segments etc. with chromatic number $\Theta(\log \log n)$.

Theorem (Krawczyk, W 2017)
There are string graphs with chromatic number $\Theta_\omega((\log \log n)^{\omega-1})$.

# Geometric intersection graphs with large chromatic number

Theorem (Burling 1965)
There are triangle-free intersection graphs of boxes in $\mathbb{R}^3$ with chromatic number $\Theta(\log\log n)$.

Theorem (Pawlik et al. 2013)
There are triangle-free intersection graphs of frames, $\mathsf{L}$-figures, segments etc. with chromatic number $\Theta(\log\log n)$.

Theorem (Krawczyk, W 2017)
There are string graphs with chromatic number $\Theta_\omega((\log\log n)^{\omega-1})$.

Are these constructions optimal? Are they “unique”?

# “Uniqueness” of the construction

Theorem (Pawlik et al. 2013)
There are triangle-free intersection graphs of frames, L-figures, segments etc. with chromatic number $\Theta(\log \log n)$.

Conjecture (Chudnovsky, Scott, Seymour 2018+)
There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every triangle-free string graph with chromatic number at least $f(k)$ contains the $k$th graph of the construction as an induced subgraph.

## “Uniqueness” of the construction

Theorem (Pawlik et al. 2013)
There are triangle-free intersection graphs of frames, $\mathsf{L}$-figures, segments etc. with chromatic number $\Theta(\log\log n)$.

Conjecture (Chudnovsky, Scott, Seymour 2018+)
There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every triangle-free string graph with chromatic number at least $f(k)$ contains the $k$th graph of the construction as an induced subgraph.

“We have little faith in this conjecture.”

## “Uniqueness” of the construction

Theorem (Pawlik et al. 2013)
There are triangle-free intersection graphs of frames, L-figures, segments etc. with chromatic number $\Theta(\log\log n)$.

Conjecture (Chudnovsky, Scott, Seymour 2018+)
There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every triangle-free string graph with chromatic number at least $f(k)$ contains the $k$th graph of the construction as an induced subgraph.

“We have little faith in this conjecture.”

This is not true for Burling’s construction of boxes in $\mathbb{R}^3$!
(Reed, Allwright 2008; Magnant, Martin 2011)

## "Uniqueness" of the construction

Theorem (Pawlik et al. 2013)
There are triangle-free intersection graphs of frames, L-figures, segments etc. with chromatic number $\Theta(\log\log n)$.

Conjecture (Chudnovsky, Scott, Seymour 2018+)
There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every triangle-free string graph with chromatic number at least $f(k)$ contains the $k$th graph of the construction as an induced subgraph.

"We have little faith in this conjecture."

This is not true for Burling's construction of boxes in $\mathbb{R}^3$!
(Reed, Allwright 2008; Magnant, Martin 2011)

Intermediate goal: Upper bounds like $O((\log\log n)^c)$

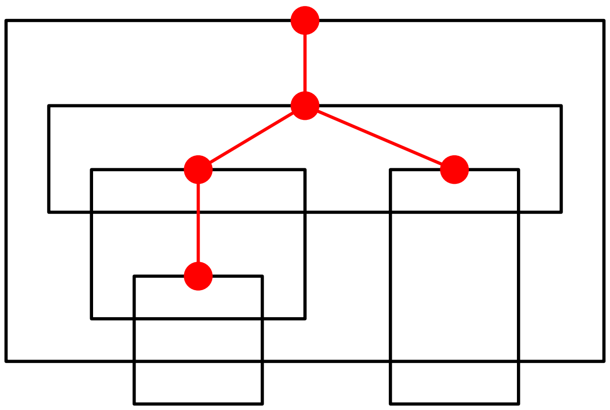
## Upper bounds on the chromatic number

Theorem (Krawczyk, Pawlik, W 2015)
Triangle-free intersection graphs of frames have chromatic number $O(\log\log n)$.

# Upper bounds on the chromatic number

Theorem (Krawczyk, Pawlik, W 2015)
Triangle-free intersection graphs of frames have chromatic number $O(\log \log n)$.

Idea: Reduce to the case of "downward" intersections.
Then, apply an on-line $O(\log \ell)$-coloring algorithm to each branch of the underlying tree, where $\ell$ is some measure of the length of the branch such that $\ell = O(\log n)$.



## Upper bounds on the chromatic number

Theorem (Krawczyk, Pawlik, W 2015)
Triangle-free intersection graphs of frames have chromatic number $O(\log\log n)$.

Theorem (Krawczyk, W 2017)
Intersection graphs of frames have chromatic number $O_\omega((\log\log n)^{\omega-1})$.

Theorem (McGuinness 1996 / Suk 2014 / Rok, W 2014)
Intersection graphs of $\mathsf{L}$-figures / segments / $x$-monotone curves have chromatic number $O_\omega(\log n)$.

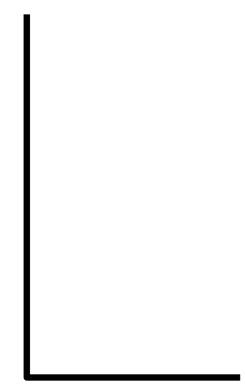
## Upper bounds on the chromatic number

**Theorem (Krawczyk, Pawlik, W 2015)**
Triangle-free intersection graphs of frames have chromatic number $O(\log\log n)$.

**Theorem (Krawczyk, W 2017)**
Intersection graphs of frames have chromatic number $O_\omega((\log\log n)^{\omega-1})$.

**Theorem (McGuinness 1996 / Suk 2014 / Rok, W 2014)**
Intersection graphs of $\mathsf{L}$-figures / segments / $x$-monotone curves have chromatic number $O_\omega(\log n)$.

**Theorem (W 2018+)**
Triangle-free intersection graphs of $\mathsf{L}$-figures have chromatic number $O(\log\log n)$.



# Upper bounds on the chromatic number

Theorem (Krawczyk, Pawlik, W 2015)
Triangle-free intersection graphs of frames have chromatic number $O(\log\log n)$.

Theorem (Krawczyk, W 2017)
Intersection graphs of frames have chromatic number $O_\omega((\log\log n)^{\omega+1})$.

Theorem (McGuinness 1996 / Suk 2014 / Rok, W 2014)
Intersection graphs of L-figures / segments / $x$-monotone curves have chromatic number $O_\omega(\log n)$.

Theorem (W 2018+)
Triangle-free intersection graphs of L-figures have chromatic number $O(\log\log n)$.

# Coloring triangle-free L-figures

Theorem (Chudnovsky, Scott, Seymour 2018+)
There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every string graph $G$ contains a vertex $v$ such that the vertices at distance $\leqslant 2$ from $v$ in $G$ have chromatic number $\geqslant \chi(G)/f(\omega(G))$.

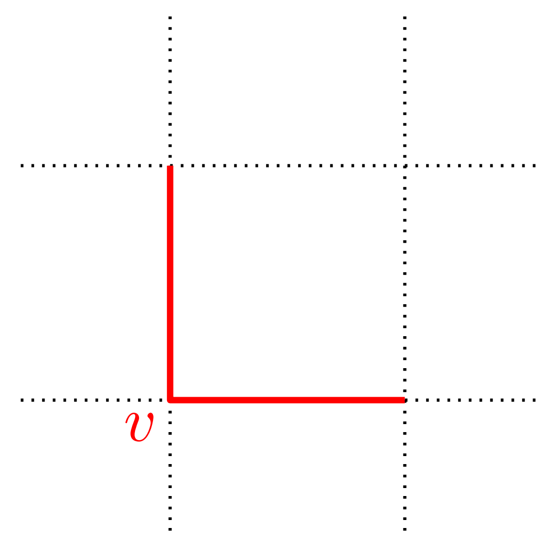
We prove that the L-figures at distance 2 from a fixed L-figure $v$ have chromatic number $O(\log\log n)$.

## Coloring triangle-free L-figures

Theorem (Chudnovsky, Scott, Seymour 2018+)
There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every string graph $G$ contains a vertex $v$ such that the vertices at distance $\leqslant 2$ from $v$ in $G$ have chromatic number $\geqslant \chi(G)/f(\omega(G))$.

We prove that the L-figures at distance 2 from a fixed L-figure $v$ have chromatic number $O(\log\log n)$.

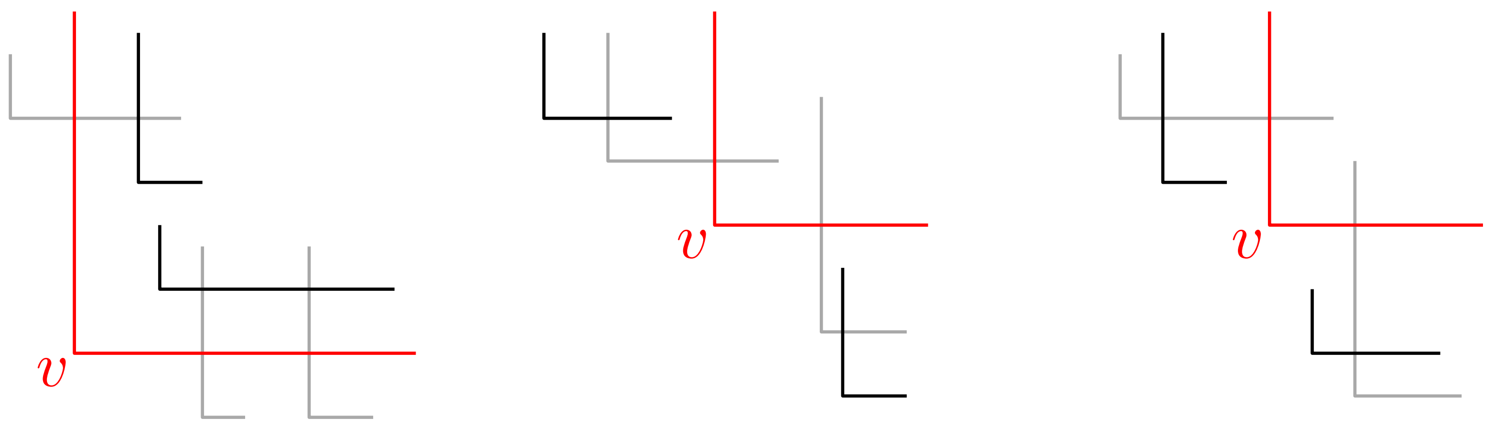
Theorem (McGuinness 1996)
The class of intersection graphs of L-figures crossing a fixed line is $\chi$-bounded.



# Coloring triangle-free L-figures

Theorem (Chudnovsky, Scott, Seymour 2018+)
There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every string graph $G$ contains a vertex $v$ such that the vertices at distance $\leqslant 2$ from $v$ in $G$ have chromatic number $\geqslant \chi(G)/f(\omega(G))$.

We prove that the L-figures at distance 2 from a fixed L-figure $v$ have chromatic number $O(\log\log n)$.

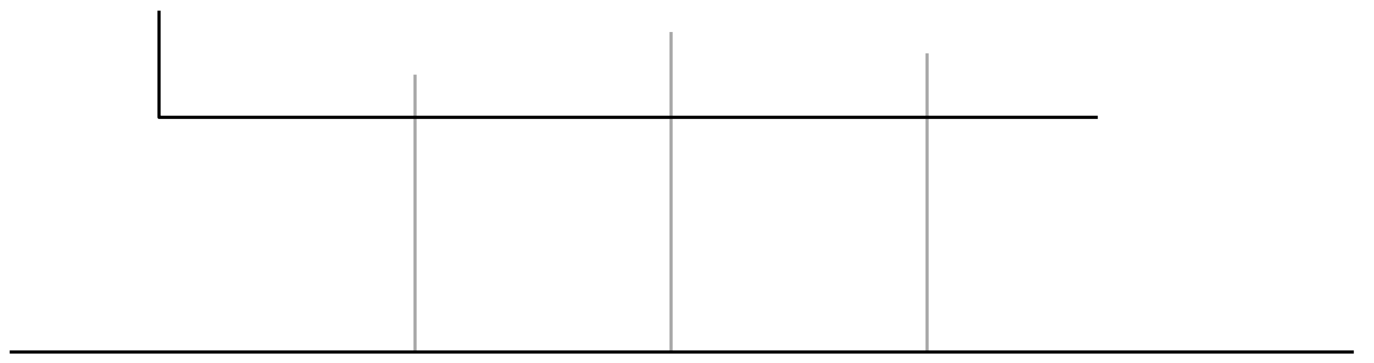



key case

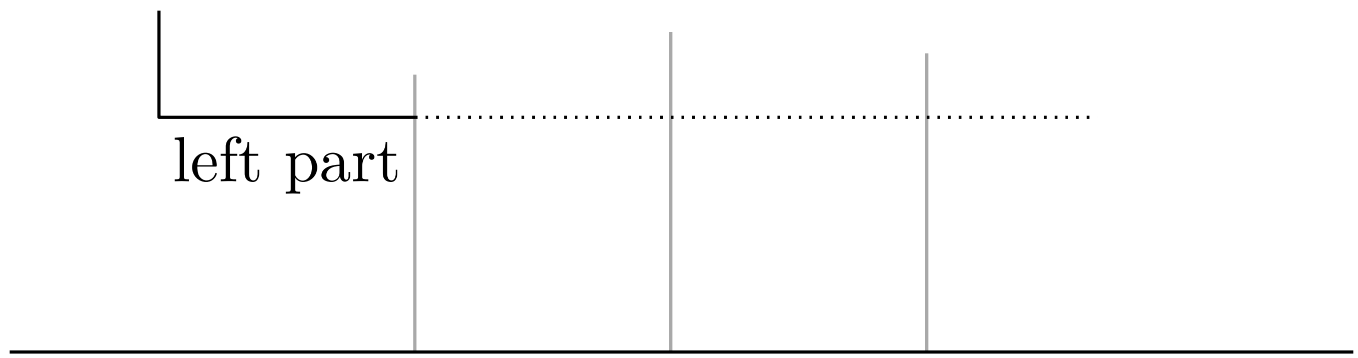
equivalent to key case

recursion + an additional trick

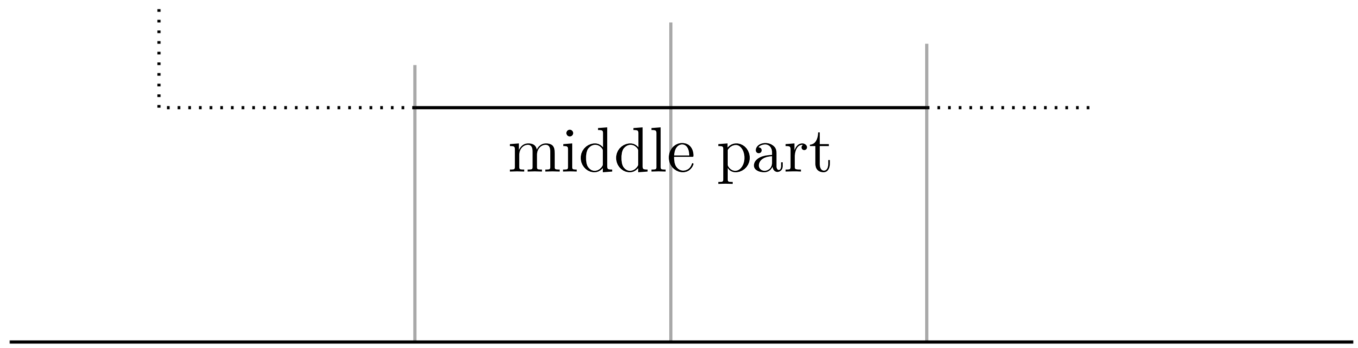
# Coloring triangle-free L-figures at distance 2, key case



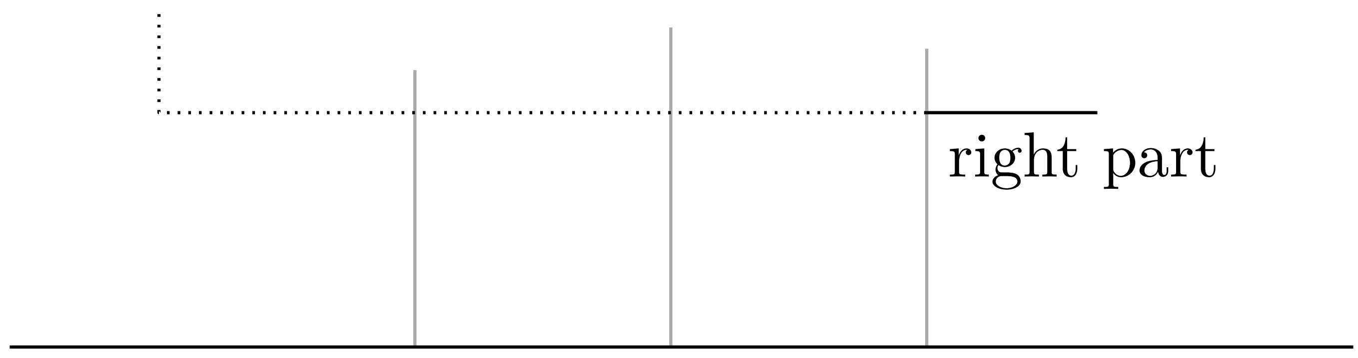
# Coloring triangle-free L-figures at distance 2, key case



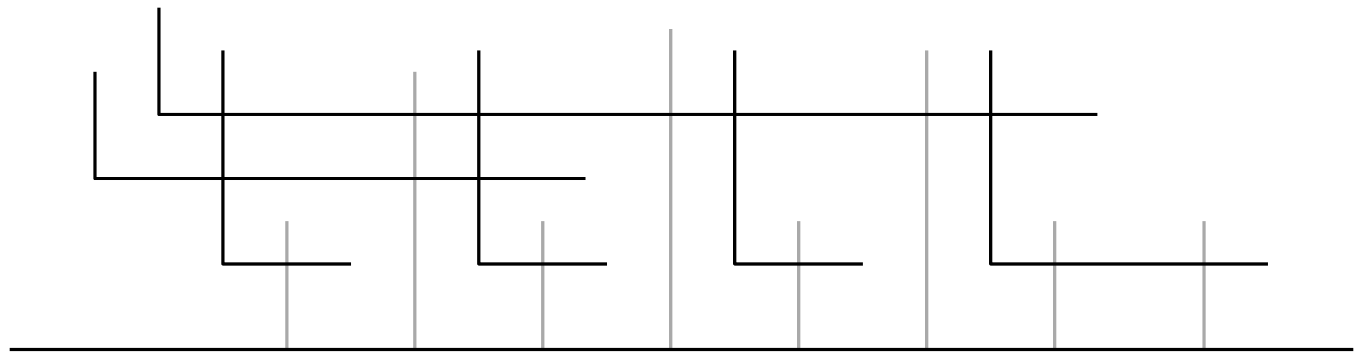
# Coloring triangle-free L-figures at distance 2, key case



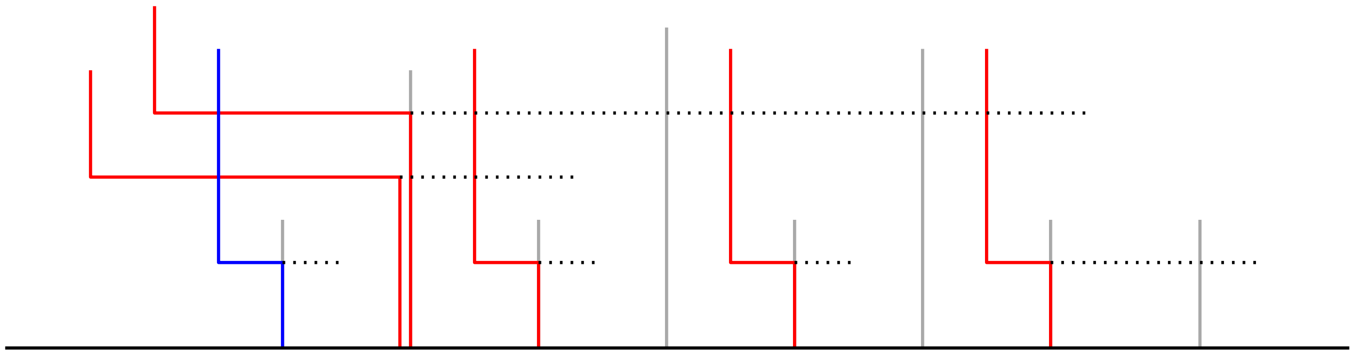
# Coloring triangle-free L-figures at distance 2, key case



# Coloring triangle-free L-figures at distance 2, key case



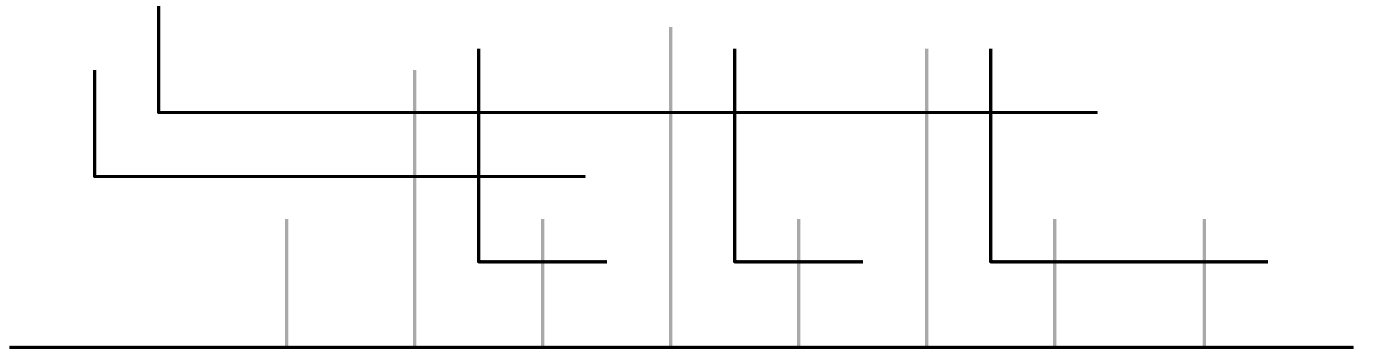
# Coloring triangle-free L-figures at distance 2, key case



1. Color to distinguish the left-left intersections

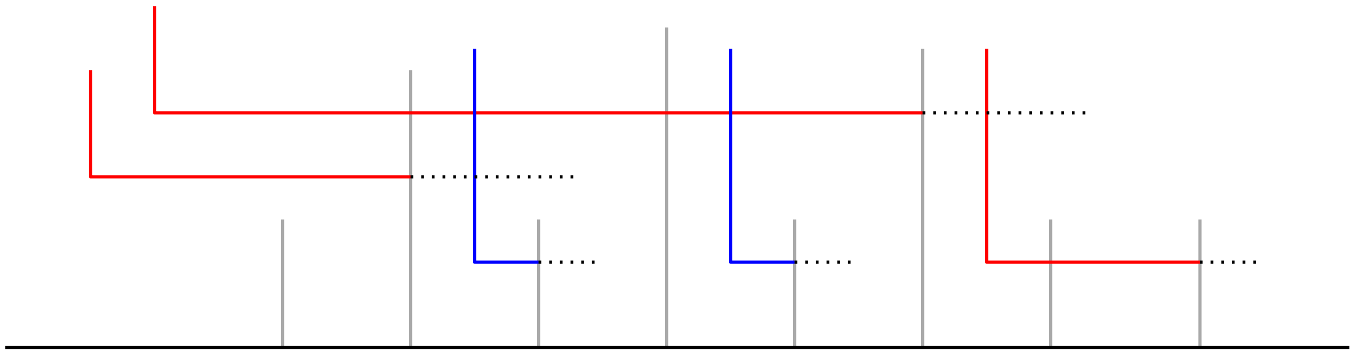
Theorem (McGuinness 2000; Suk 2014; Rok, W 2014)
The class of intersection graphs of grounded curves is $\chi$-bounded.

# Coloring triangle-free L-figures at distance 2, key case



1. Color to distinguish the left-left intersections

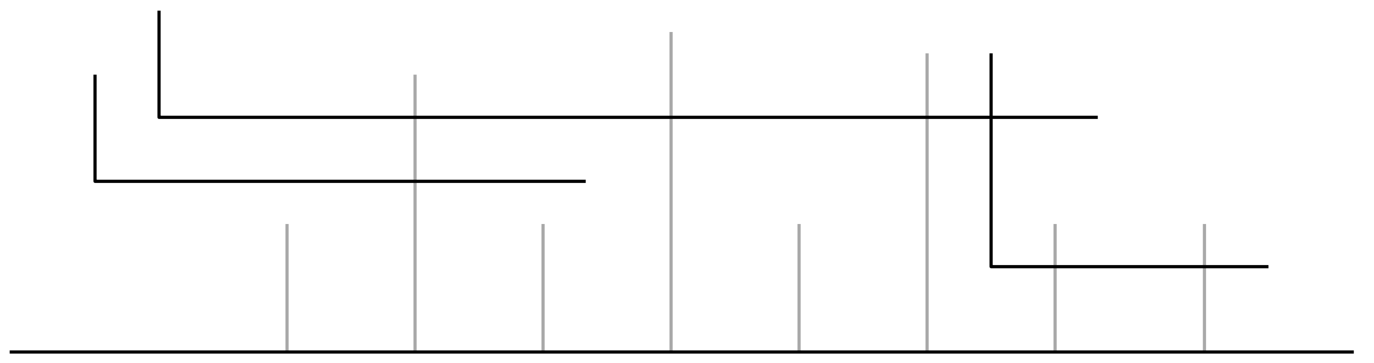
# Coloring triangle-free L-figures at distance 2, key case



1. Color to distinguish the left-left intersections
2. Color to distinguish the left-middle intersections

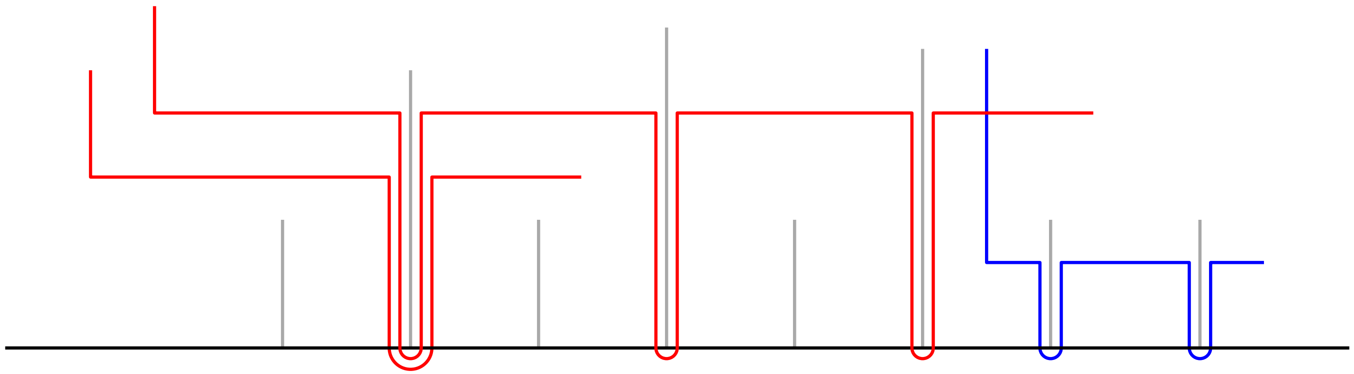
We will show how to do this using $O(\log\log n)$ colors.

# Coloring triangle-free L-figures at distance 2, key case



1. Color to distinguish the left-left intersections
2. Color to distinguish the left-middle intersections

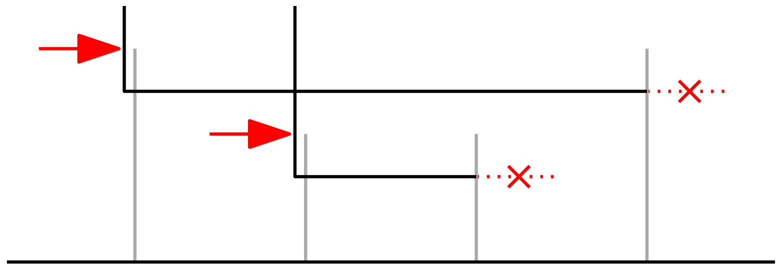
# Coloring triangle-free L-figures at distance 2, key case



1. Color to distinguish the left-left intersections
2. Color to distinguish the left-middle intersections
3. Color to distinguish the left-right intersections

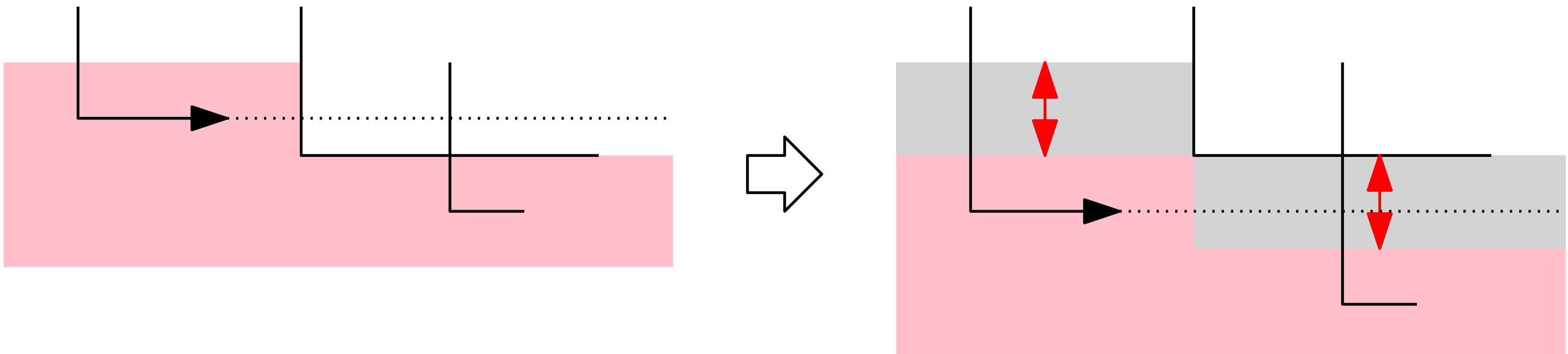
Theorem (Rok, W 2017)
The class of intersection graphs of multi-grounded curves, where only the left-most and the right-most upper parts of the curves are allowed to intersect, is $\chi$-bounded.

# Coloring to distinguish the left-middle intersections

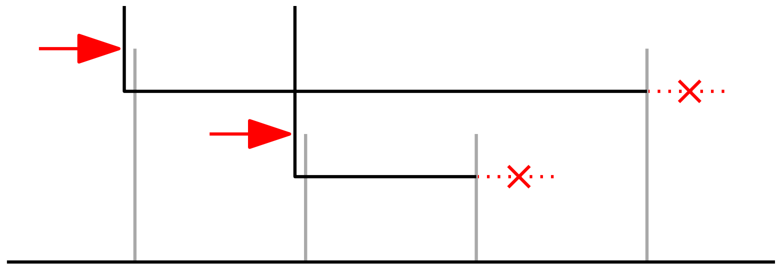


Assumptions:

1. The right parts are empty
2. The left parts are pushed to the right as far as possible
3. There are no *extension blockers*



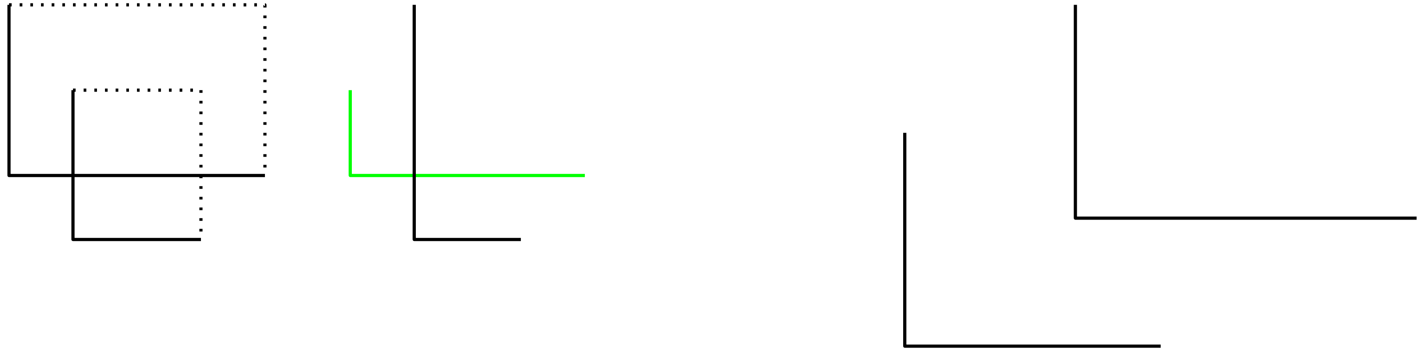
# Coloring to distinguish the left-middle intersections



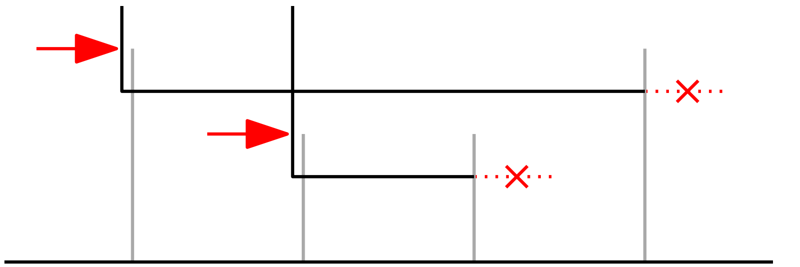
Assumptions:

1. The right parts are empty
2. The left parts are pushed to the right as far as possible
3. There are no *extension blockers*

We use a special color green on L-figures whose vertical legs intersect no other L-figures (including the green ones).
We try to "close" the remaining L-figures into frames.

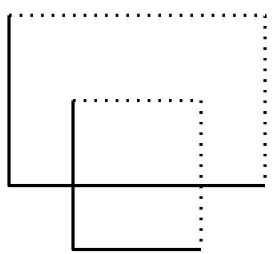


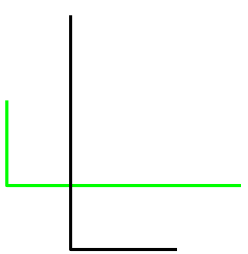
# Coloring to distinguish the left-middle intersections

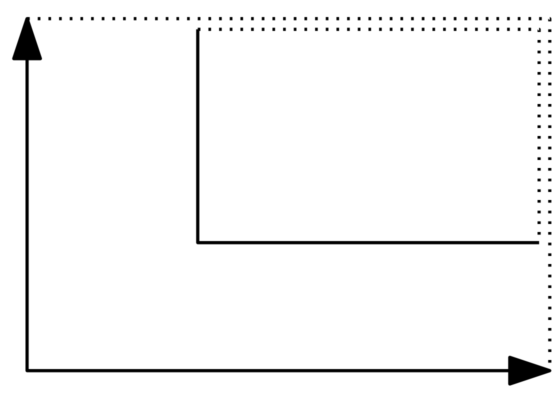


Assumptions:
1. The right parts are empty
2. The left parts are pushed to the right as far as possible
3. There are no *extension blockers*

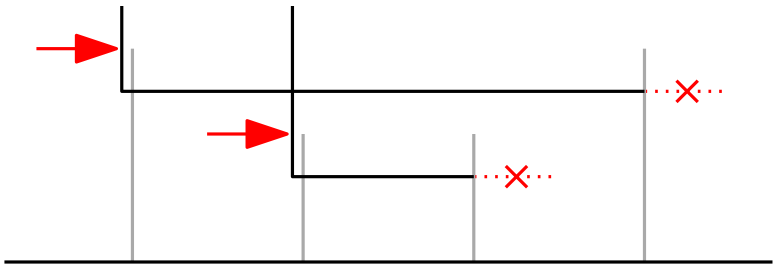
We use a special color green on L-figures whose vertical legs intersect no other L-figures (including the green ones).
We try to "close" the remaining L-figures into frames.





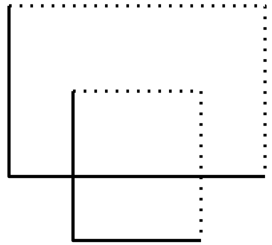


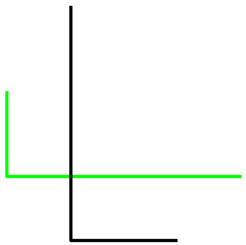
# Coloring to distinguish the left-middle intersections

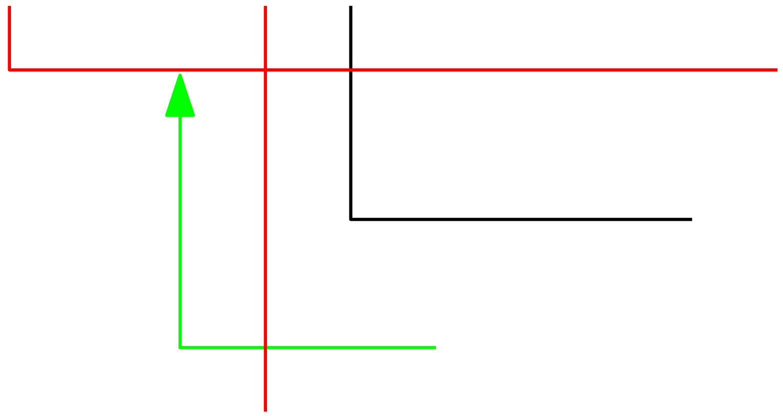


Assumptions:
1. The right parts are empty
2. The left parts are pushed to the right as far as possible
3. There are no *extension blockers*

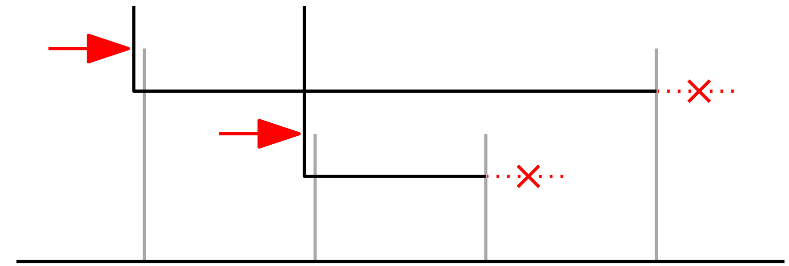
We use a special color green on L-figures whose vertical legs intersect no other L-figures (including the green ones).
We try to "close" the remaining L-figures into frames.





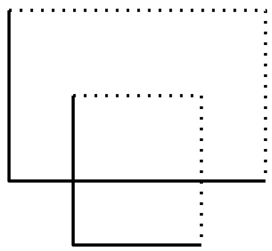


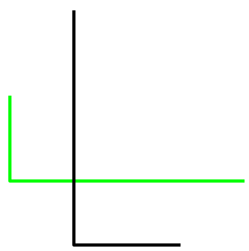
# Coloring to distinguish the left-middle intersections

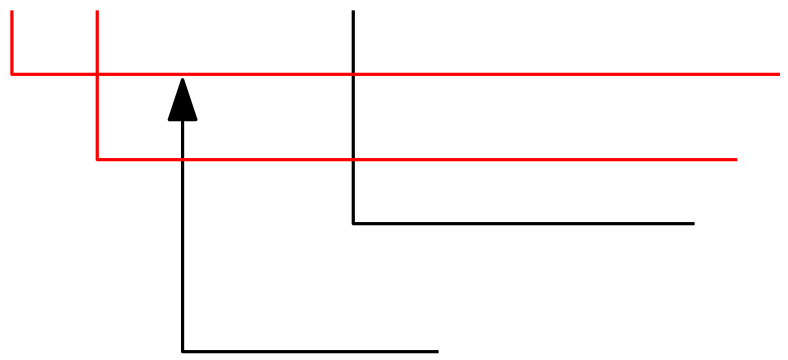


Assumptions:
1. The right parts are empty
2. The left parts are pushed to the right as far as possible
3. There are no *extension blockers*

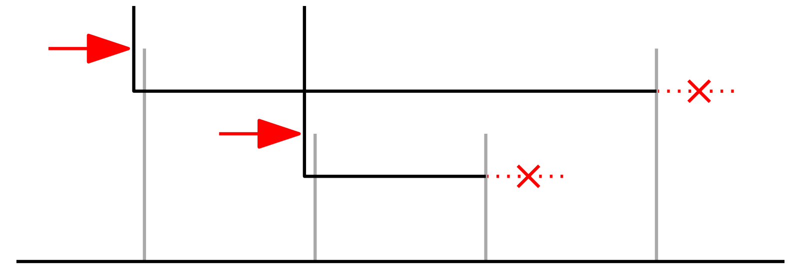
We use a special color green on L-figures whose vertical legs intersect no other L-figures (including the green ones).
We try to "close" the remaining L-figures into frames.







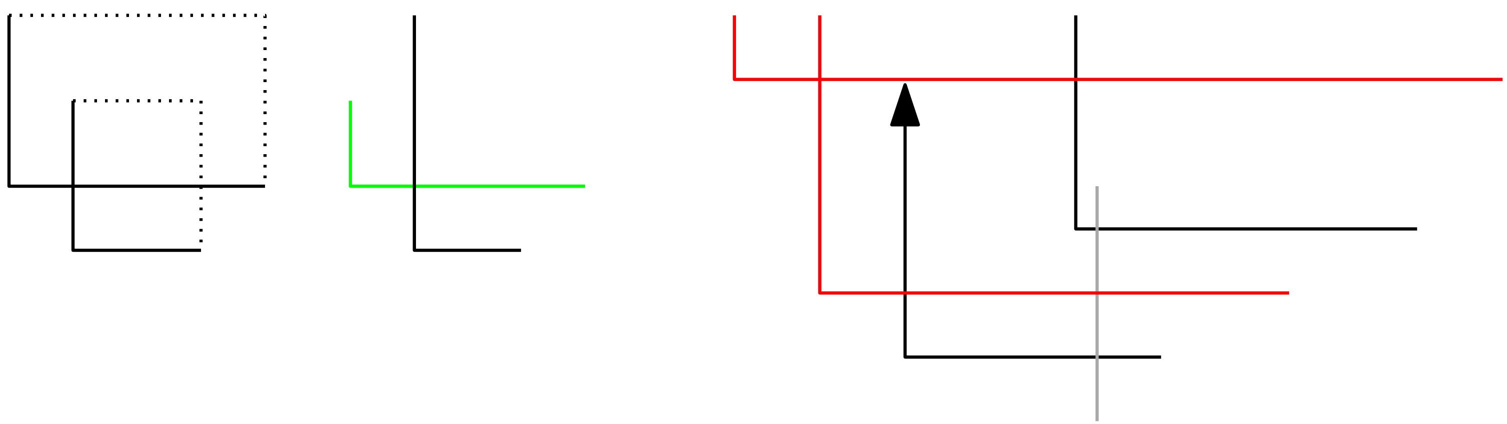
# Coloring to distinguish the left-middle intersections



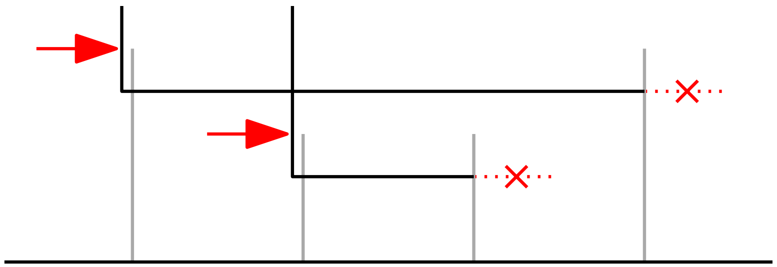
Assumptions:

1. The right parts are empty
2. The left parts are pushed to the right as far as possible
3. There are no *extension blockers*

We use a special color green on L-figures whose vertical legs intersect no other L-figures (including the green ones).
We try to "close" the remaining L-figures into frames.



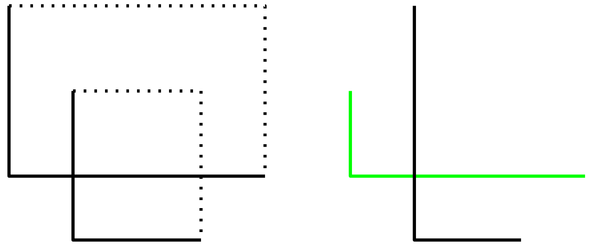
# Coloring to distinguish the left-middle intersections

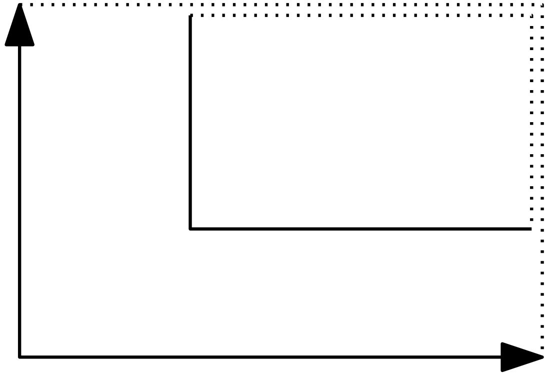


Assumptions:

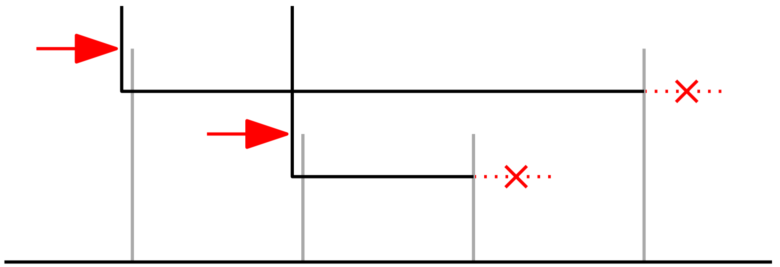
1. The right parts are empty
2. The left parts are pushed to the right as far as possible
3. There are no *extension blockers*

We use a special color green on L-figures whose vertical legs intersect no other L-figures (including the green ones).
We try to “close” the remaining L-figures into frames.





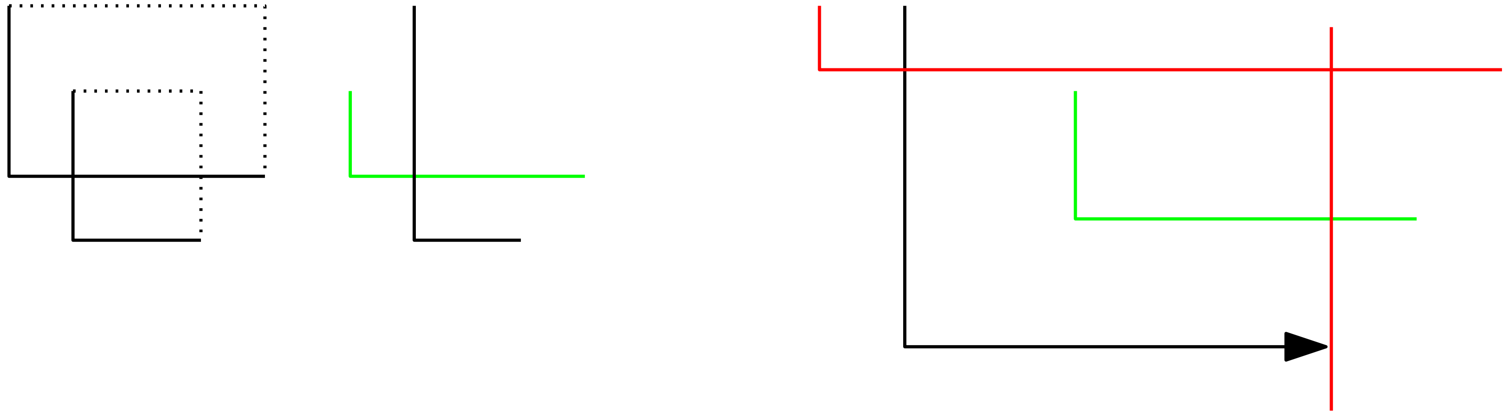
# Coloring to distinguish the left-middle intersections



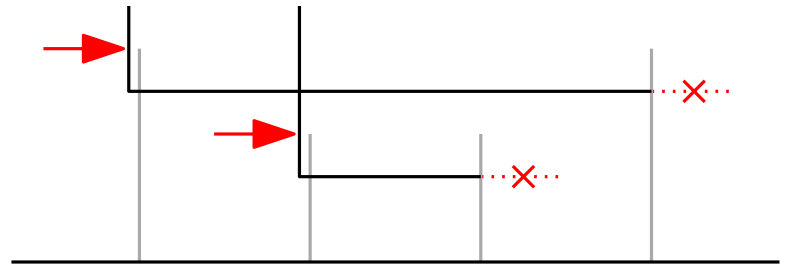
Assumptions:

1. The right parts are empty
2. The left parts are pushed to the right as far as possible
3. There are no *extension blockers*

We use a special color green on L-figures whose vertical legs intersect no other L-figures (including the green ones).
We try to "close" the remaining L-figures into frames.

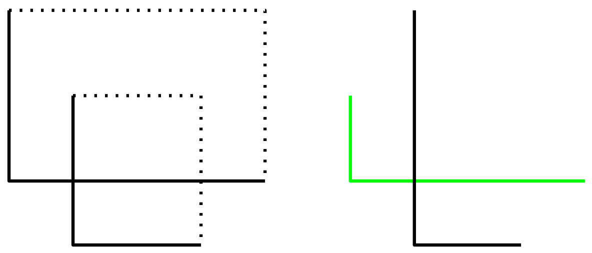


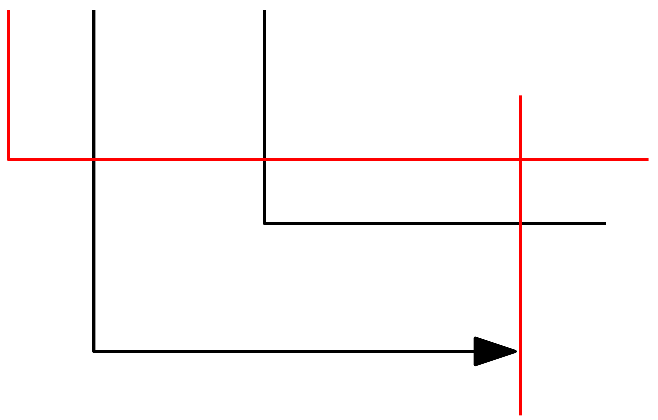
# Coloring to distinguish the left-middle intersections



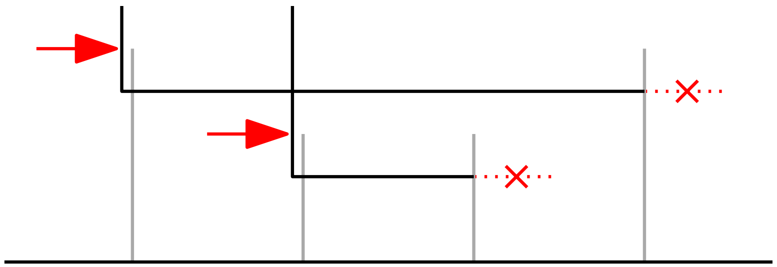
Assumptions:
1. The right parts are empty
2. The left parts are pushed to the right as far as possible
3. There are no *extension blockers*

We use a special color green on L-figures whose vertical legs intersect no other L-figures (including the green ones).
We try to "close" the remaining L-figures into frames.





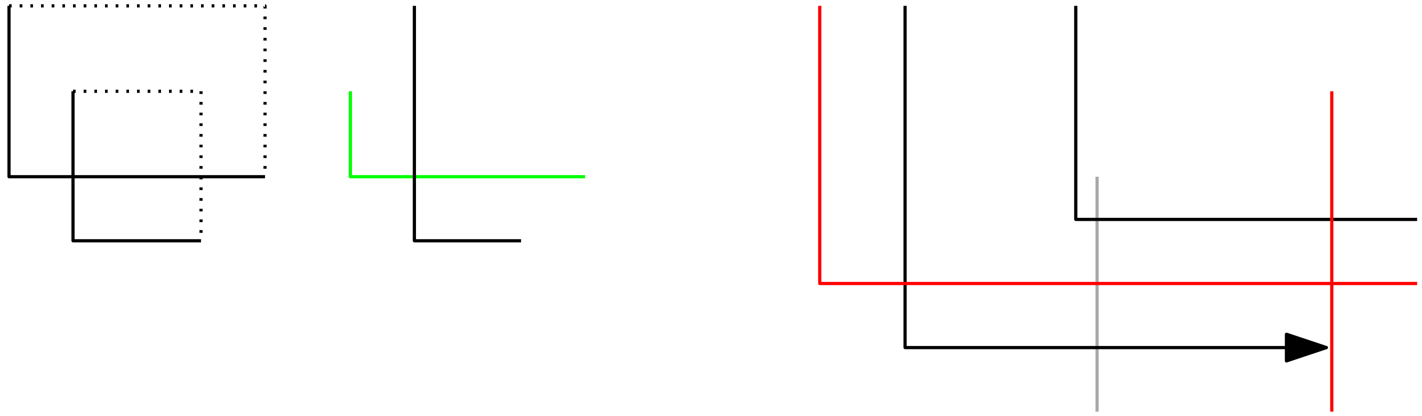
# Coloring to distinguish the left-middle intersections



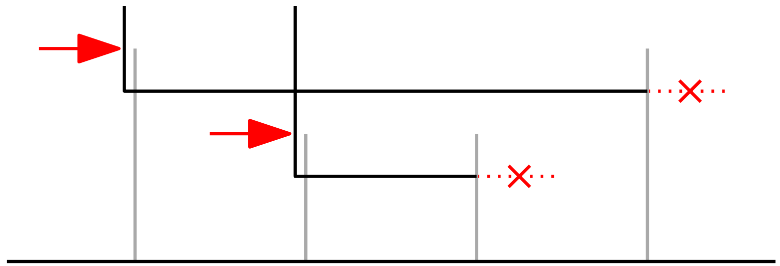
Assumptions:

1. The right parts are empty
2. The left parts are pushed to the right as far as possible
3. There are no *extension blockers*

We use a special color green on L-figures whose vertical legs intersect no other L-figures (including the green ones).
We try to "close" the remaining L-figures into frames.



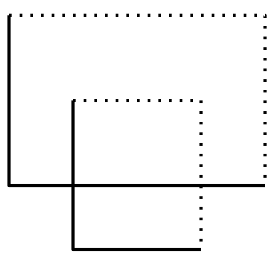
# Coloring to distinguish the left-middle intersections

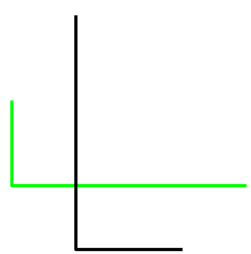


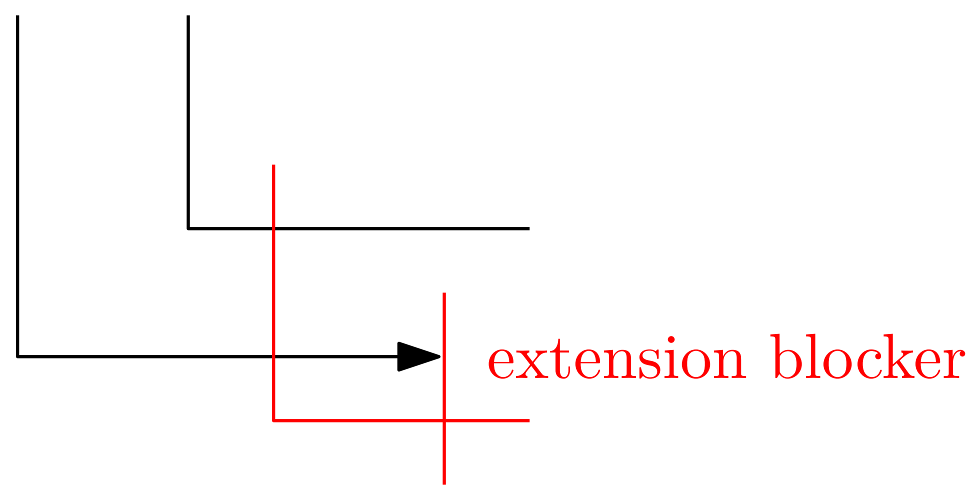
Assumptions:

1. The right parts are empty
2. The left parts are pushed to the right as far as possible
3. There are no *extension blockers*

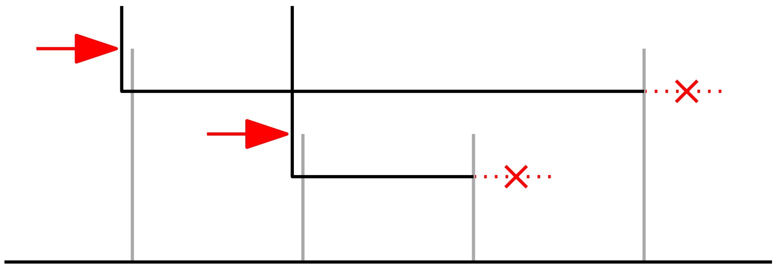
We use a special color green on L-figures whose vertical legs intersect no other L-figures (including the green ones).
We try to "close" the remaining L-figures into frames.







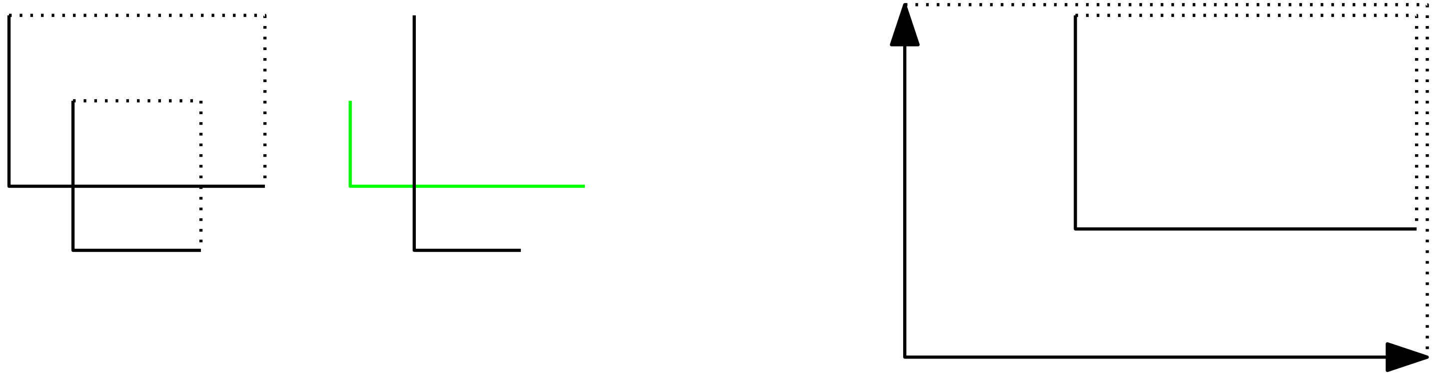
# Coloring to distinguish the left-middle intersections



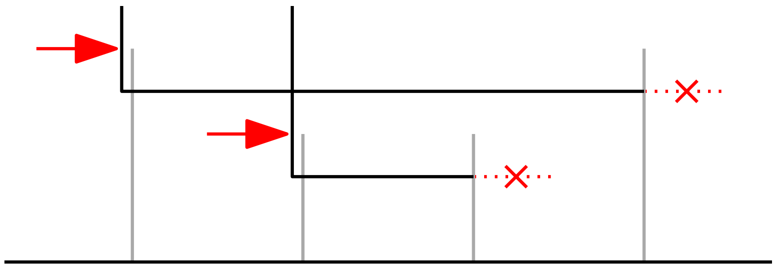
Assumptions:

1. The right parts are empty
2. The left parts are pushed to the right as far as possible
3. There are no *extension blockers*

We use a special color green on L-figures whose vertical legs intersect no other L-figures (including the green ones).
We try to "close" the remaining L-figures into frames.



# Coloring to distinguish the left-middle intersections



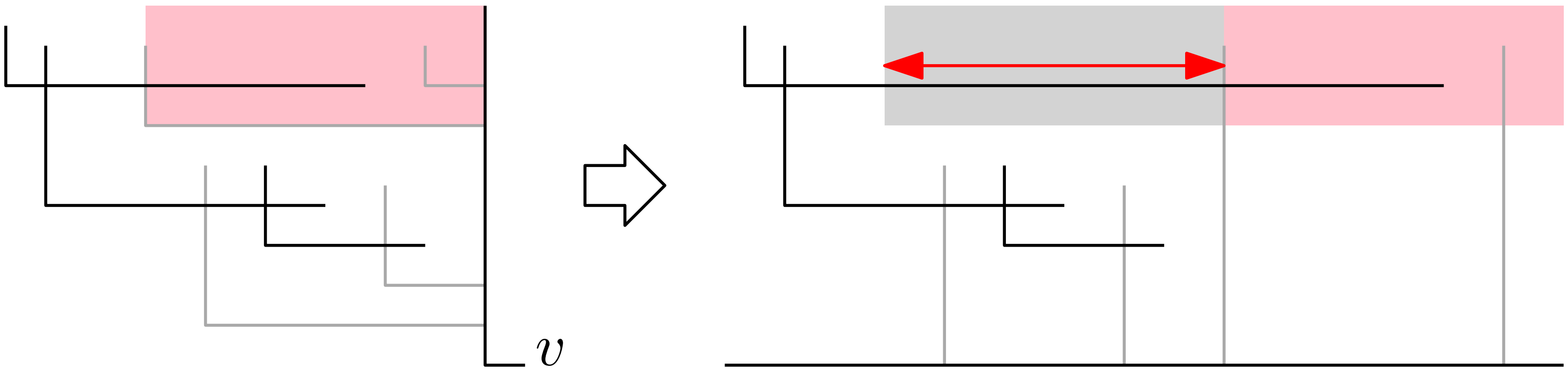
Assumptions:
1. The right parts are empty
2. The left parts are pushed to the right as far as possible
3. There are no *extension blockers*

We use a special color green on L-figures whose vertical legs intersect no other L-figures (including the green ones).
We try to "close" the remaining L-figures into frames.

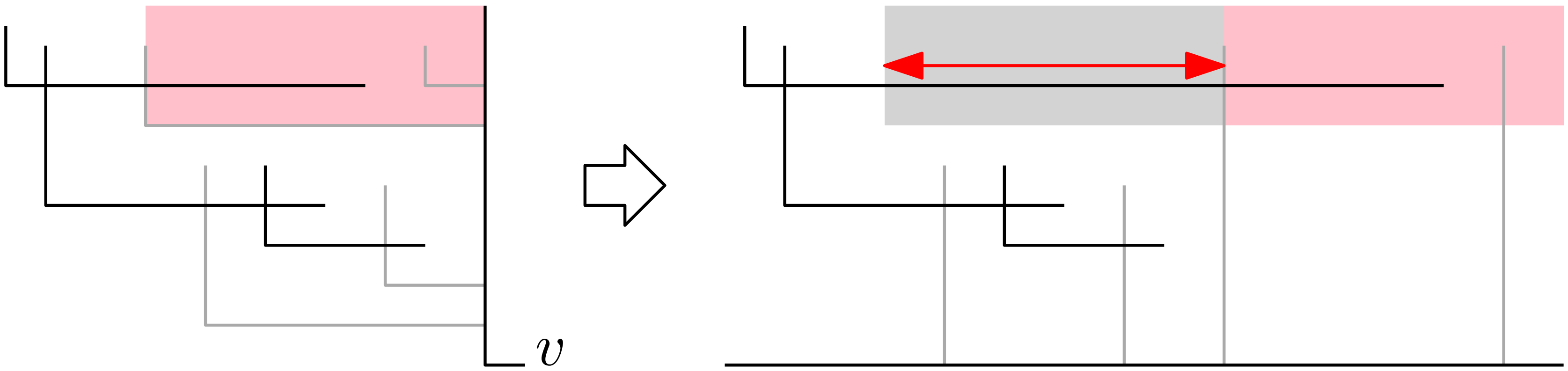
We end up with a downward-directed family of frames.

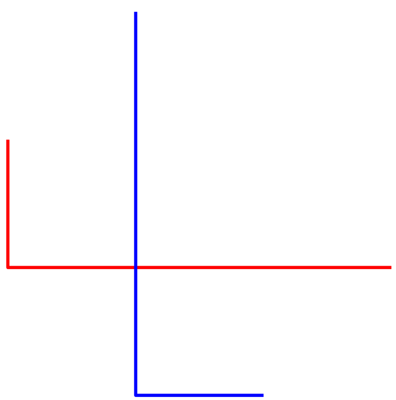
Theorem (Krawczyk, Pawlik, W 2015)
Triangle-free intersection graphs of frames have chromatic number $O(\log \log n)$.

# Coloring triangle-free L-figures at distance 2, other cases

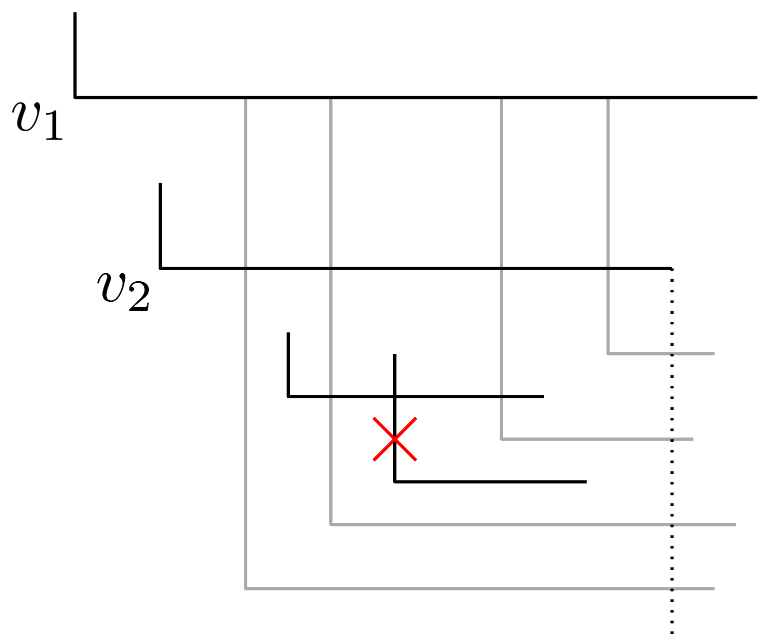


# Coloring triangle-free L-figures at distance 2, other cases



initial coloring of
all L-figures




$\chi \leqslant 4$

# Generalizations?

Theorem (W 2018+)
Triangle-free intersection graphs of $\mathsf{L}$-figures have chromatic number $O(\log\log n)$.

1. Generalization to higher clique number — ???
2. Extension to other kinds of figures — some ideas

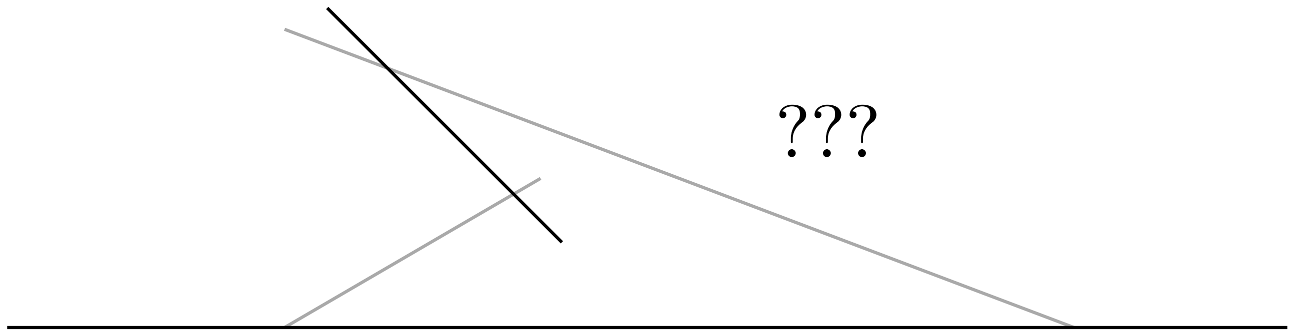
## Generalizations?

Theorem (W 2018+)
Triangle-free intersection graphs of $\mathsf{L}$-figures have chromatic number $O(\log\log n)$.

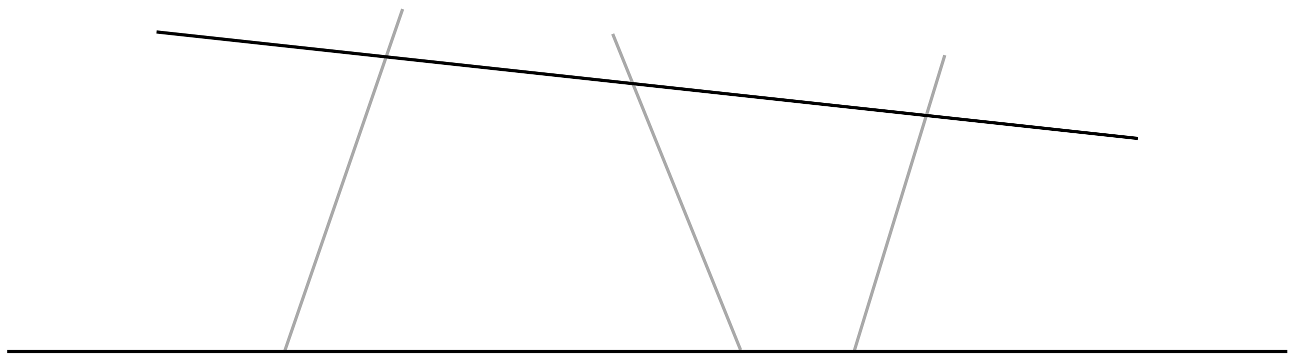
1. Generalization to higher clique number — ???
2. Extension to other kinds of figures — some ideas

Again, it suffices to bound the chromatic number of the segments at distance 2 from a fixed segment $v$.

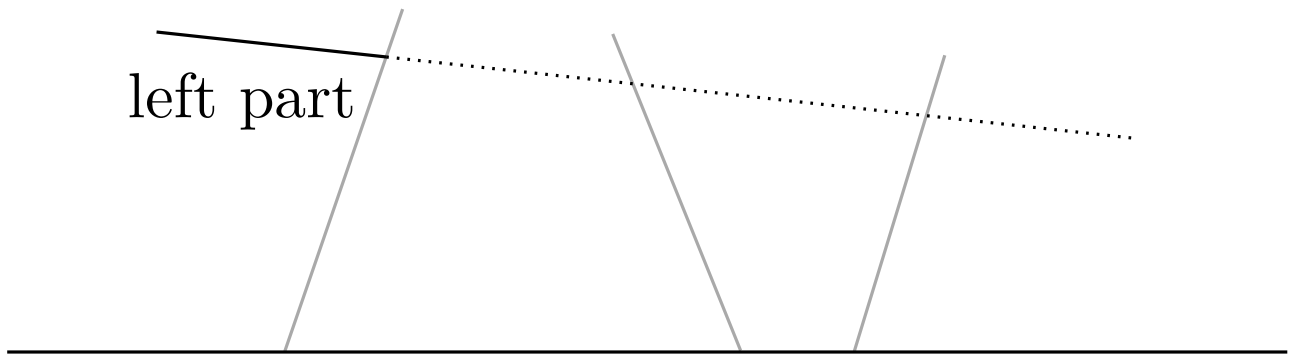
# Coloring triangle-free segments at distance 2



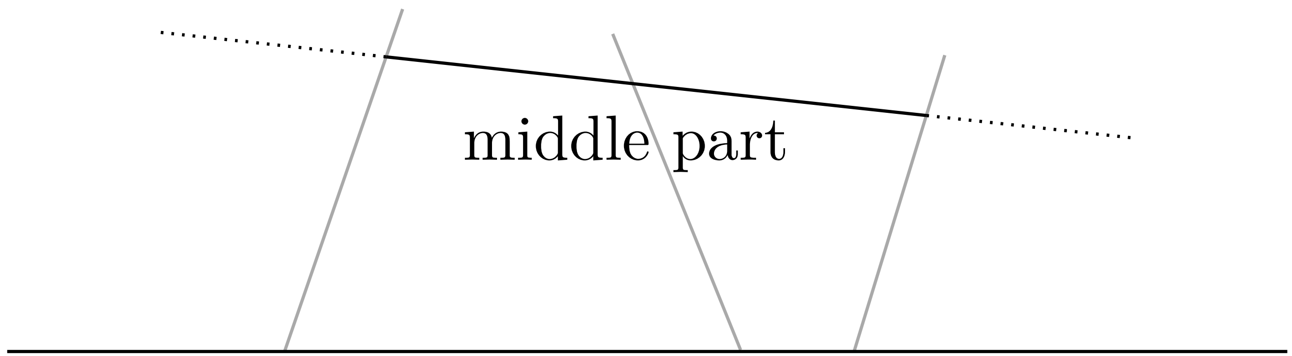
# Coloring triangle-free segments at distance 2



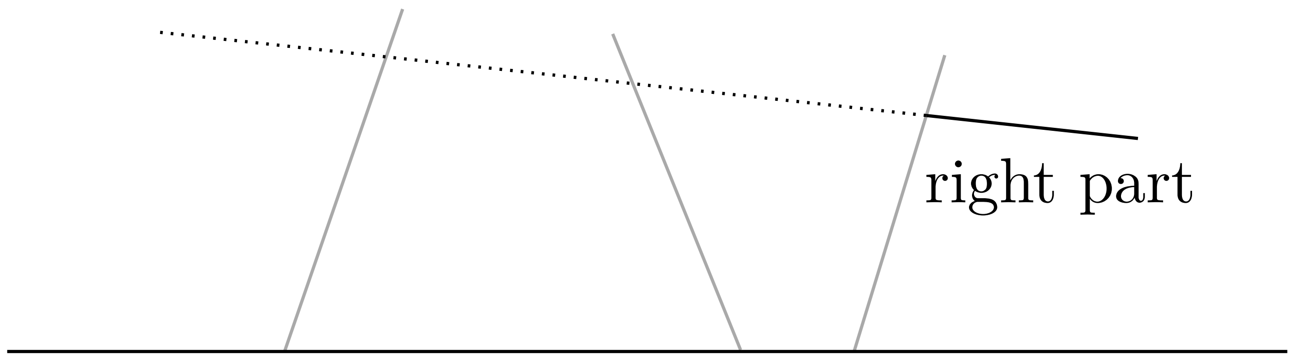
# Coloring triangle-free segments at distance 2



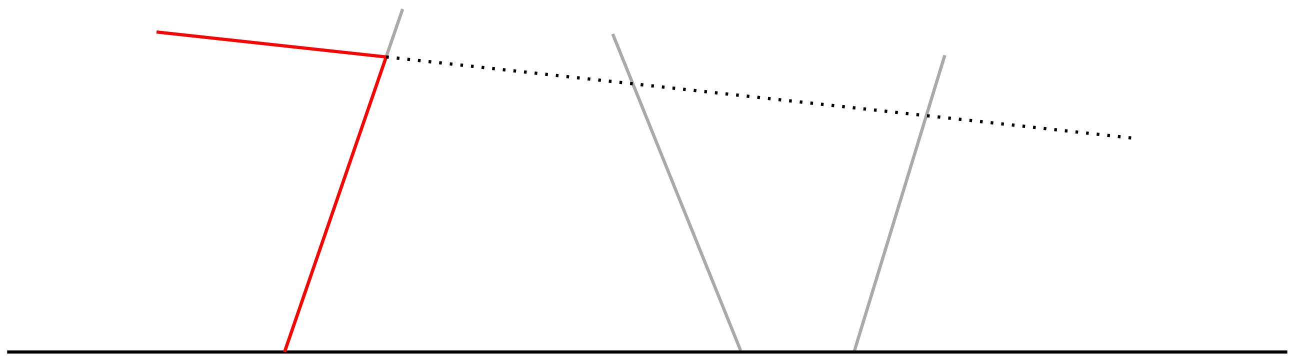
# Coloring triangle-free segments at distance 2



# Coloring triangle-free segments at distance 2

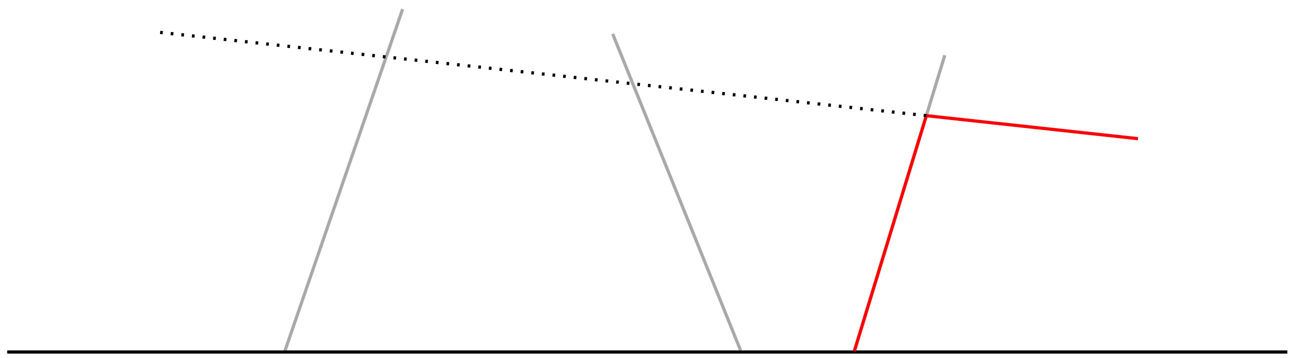


# Coloring triangle-free segments at distance 2



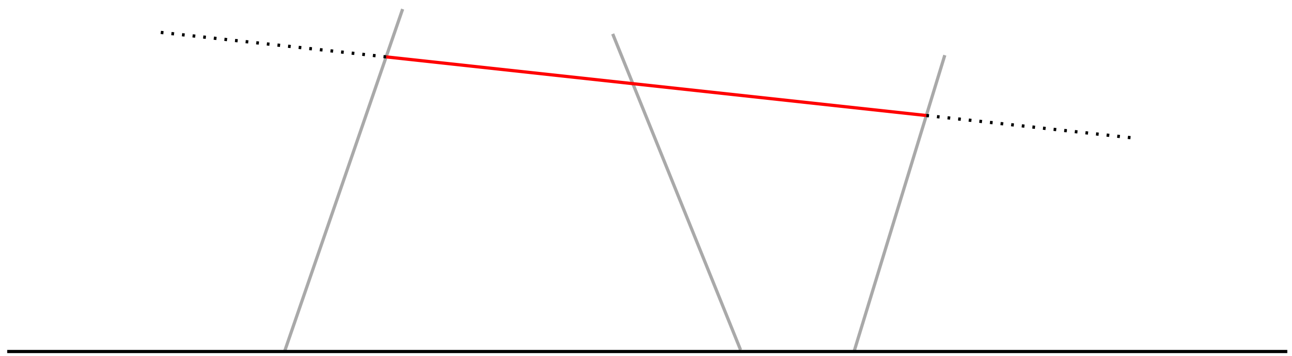
1. Distinguishing left-left intersections — as before

# Coloring triangle-free segments at distance 2



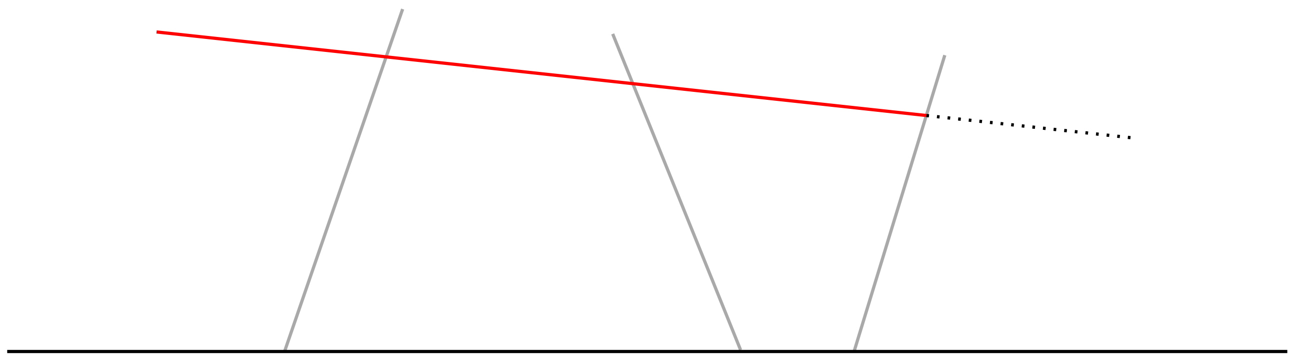
1. Distinguishing left-left intersections — as before
2. Distinguishing right-right intersections — analogously

# Coloring triangle-free segments at distance 2

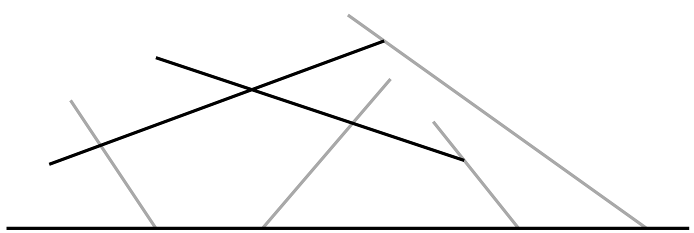


1. Distinguishing left-left intersections — as before
2. Distinguishing right-right intersections — analogously
3. Distinguishing middle-middle intersections — ???

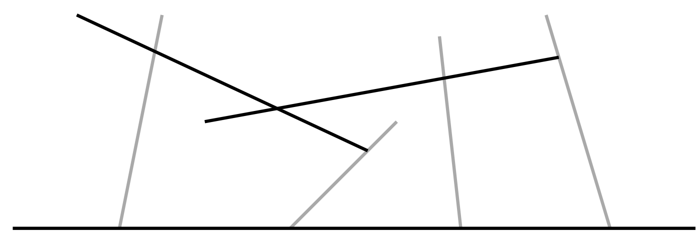
# Coloring triangle-free segments at distance 2



1. Distinguishing left-left intersections — as before
2. Distinguishing right-right intersections — analogously
3. Distinguishing middle-middle intersections — ???
4. Distinguishing left-middle intersections

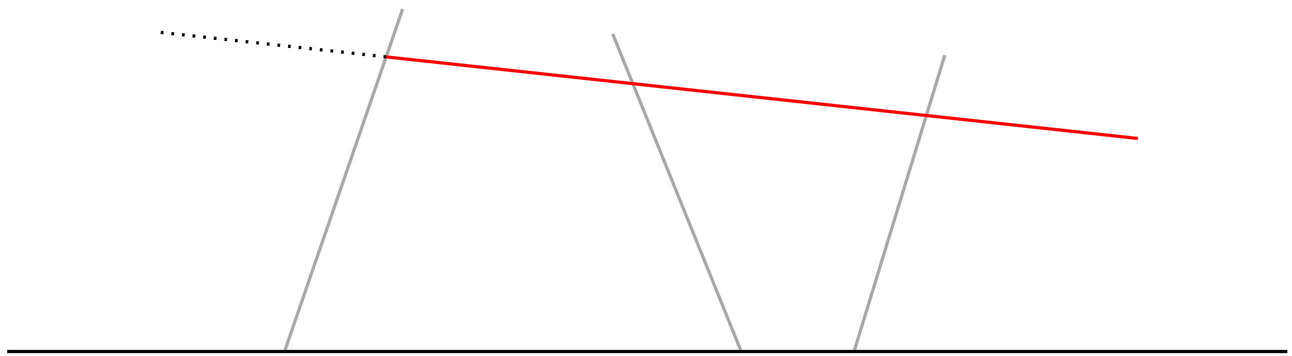


as before (!)



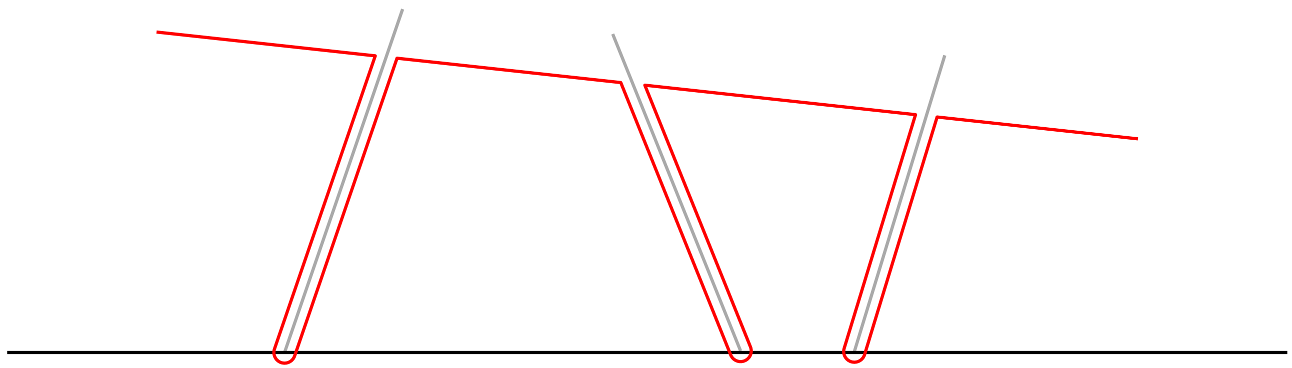
???

# Coloring triangle-free segments at distance 2



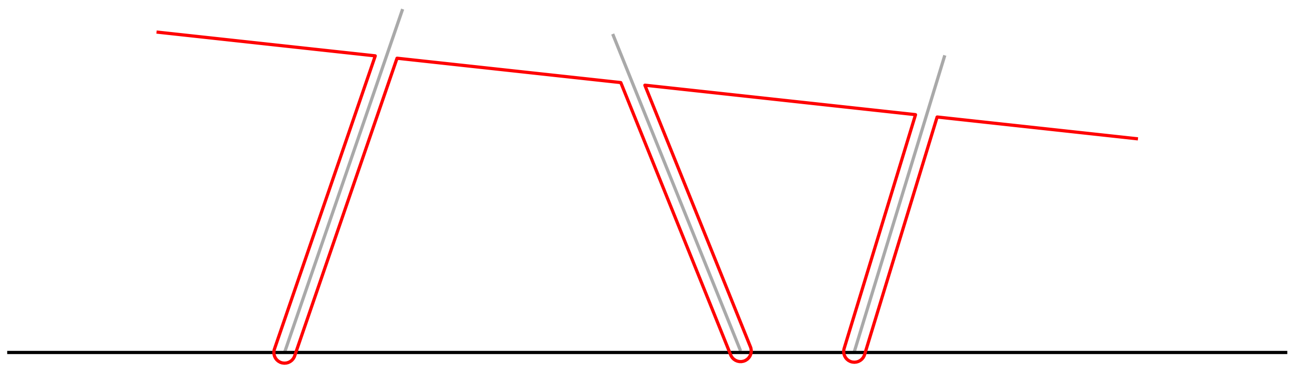
1. Distinguishing left-left intersections — as before
2. Distinguishing right-right intersections — analogously
3. Distinguishing middle-middle intersections — ???
4. Distinguishing left-middle intersections
5. Distinguishing middle-right intersections — analogously

# Coloring triangle-free segments at distance 2



1. Distinguishing left-left intersections — as before
2. Distinguishing right-right intersections — analogously
3. Distinguishing middle-middle intersections — ???
4. Distinguishing left-middle intersections
5. Distinguishing middle-right intersections — analogously
6. Distinguishing left-right intersections — as before

# Coloring triangle-free segments at distance 2



1. Distinguishing left-left intersections — as before
2. Distinguishing right-right intersections — analogously
3. Distinguishing middle-middle intersections — ???
4. Distinguishing left-middle intersections
5. Distinguishing middle-right intersections — analogously
6. Distinguishing left-right intersections — as before

This approach, if successful, can lead to an upper bound of the form $\chi = O((\log\log n)^c)$ for some *large* constant $c$.
Any ideas how to approach the bound $\chi = O(\log\log n)$?