Exchange operations on noncrossing spanning trees

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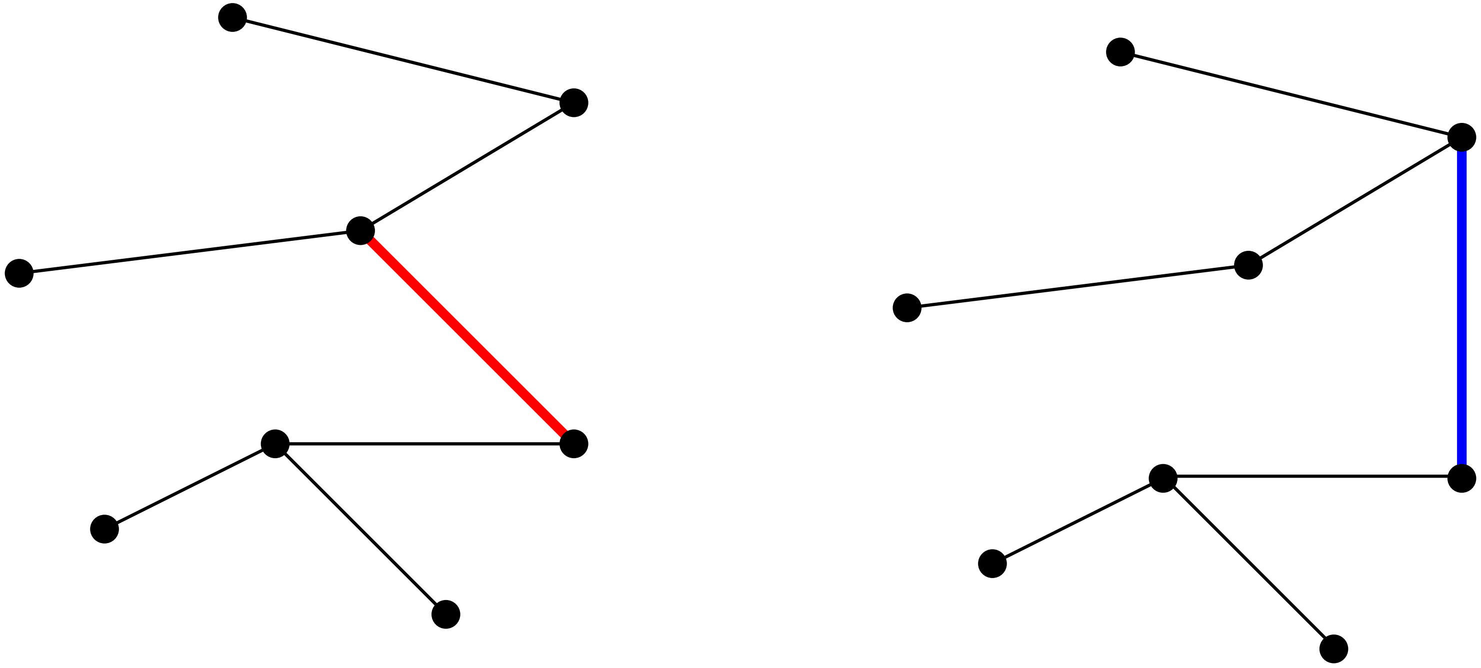


Spanning Trees — Elementary Operations

abstract spanning tree = connected graph on n vertices that does not contain cycles.

There are n^{n-2} spanning trees on n labeled vertices [Cayley, 1889]

Exchange property for graphic maroids:
If $T_1 = (V, E_1)$ and $T_2 = (V, E_2)$ are spanning trees,
$\forall e_1 \in E_1 \ \exists e_2 \in E_2 : (V, E_1 - e_1 + e_2)$ is a spanning tree.

For $n \geq 4$, there exist two edge-disjoint spanning trees.
So the diameter of the *exchange graph* equals $n - 1$.

Spanning Trees — Elementary Operations

plane spanning tree = a straight-line spanning tree on n points in the plane, no two edges cross.

S = set of n points in general position in $\mathbb{R}^2$,
$\mathcal{T}(S)$ = set of plane spanning trees on S.

For $|S| = n$,

$$\Omega(12.54^n) \leq \max_{|S|=n} |\mathcal{T}(S)| \leq O(141.07^n).$$

[Huemer and de Mier, 2015; Hoffmann et al. 2013]

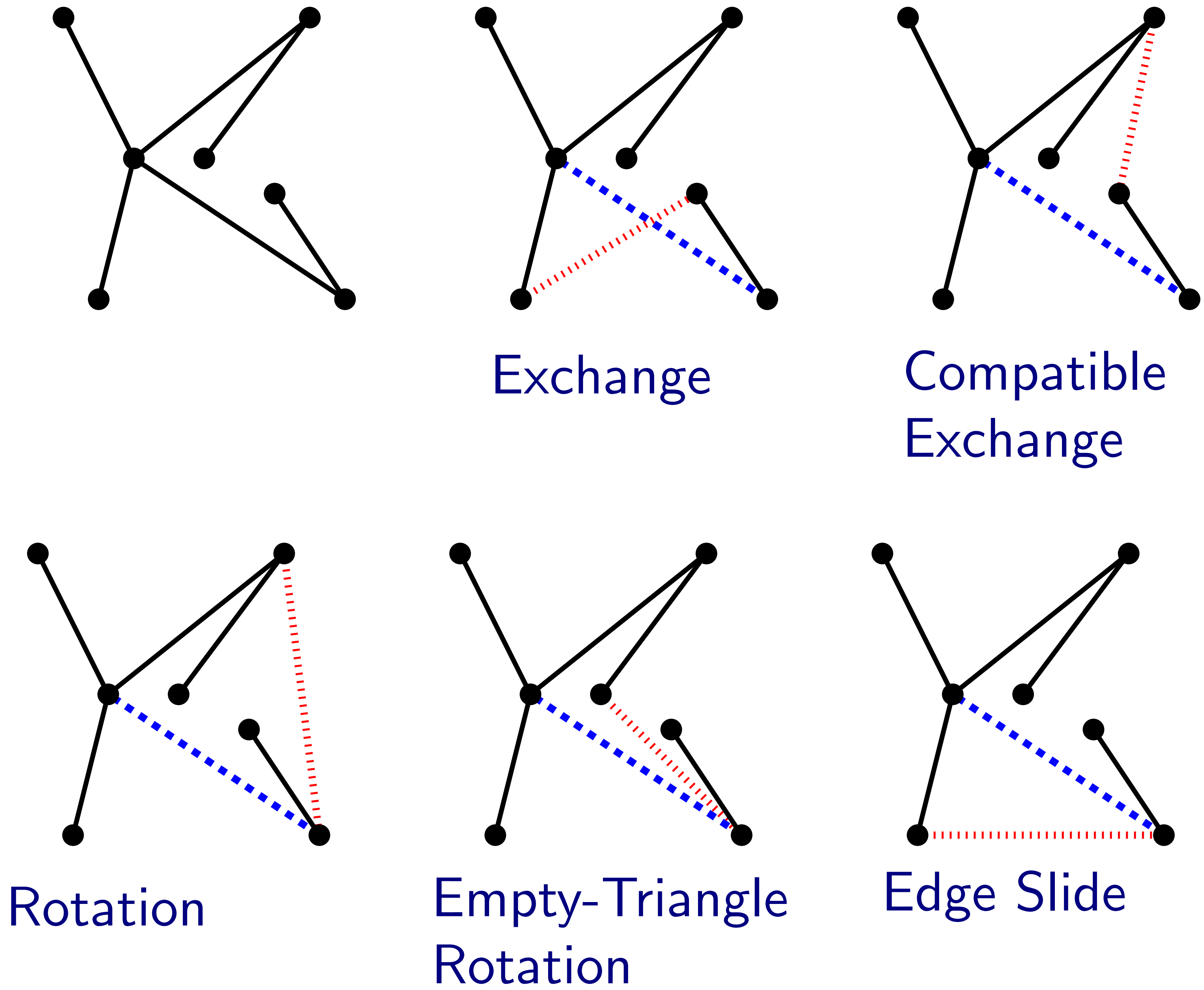
- The matroid exchange may introduce crossings!
- We restrict exchanges to plane spanning trees.

Spanning Trees — Elementary Operations

Let $T_1 = (S, E_1)$ and $T_2 = (S, E_2)$ be two trees in $\mathcal{T}(S)$.
The operation that replaces T_1 by T_2 is

- an **exchange** if there are edges e_1 and e_2 such that $E_1 \setminus E_2 = \{e_1\}$ and $E_2 \setminus E_1 = \{e_2\}$ (i.e., delete an edge e_1 from E_1 and insert a new edge e_2).
- A **compatible exchange** is an exchange such that the graph $(S, E_1 \cup E_2)$ is a noncrossing straight-line graph (i.e., e_1 and e_2 do not cross).
- A **rotation** is a compatible exchange such that e_1 and e_2 have a common endpoint $p = e_1 \cap e_2$.
- An **empty-triangle rotation** is a rotation such that the edges of neither T_1 nor T_2 intersect the interior of the triangle $\Delta(pqr)$ formed by the vertices of e_1 and e_2.
- An **edge slide** is an empty-triangle rotation such that $qr \in E_1 \cap E_2$.

Spanning Trees — Elementary Operations



Spanning Trees — Elementary Operations

All five operations define *connected* transition graphs for every point set in general position.

Operation	Single Operation Upper Bound	Single Operation Lower Bound
Exchange	$2n-4$	$\lfloor \frac{3n}{2} \rfloor - 5$ [HHM+99]
Compatible Ex.	$2n-4$	$\lfloor \frac{3n}{2} \rfloor - 5$
Rotation	$2n-4$ [AF96]	$\lfloor \frac{3n}{2} \rfloor - 4$
Empty-Tri. Rot.	$O(n \log n)$	$\lfloor \frac{3n}{2} \rfloor - 4$
Edge Slide	$O(n^2)$ [AR07]	$\Omega(n^2)$ [AR07]

Current upper and lower bounds for the diameter

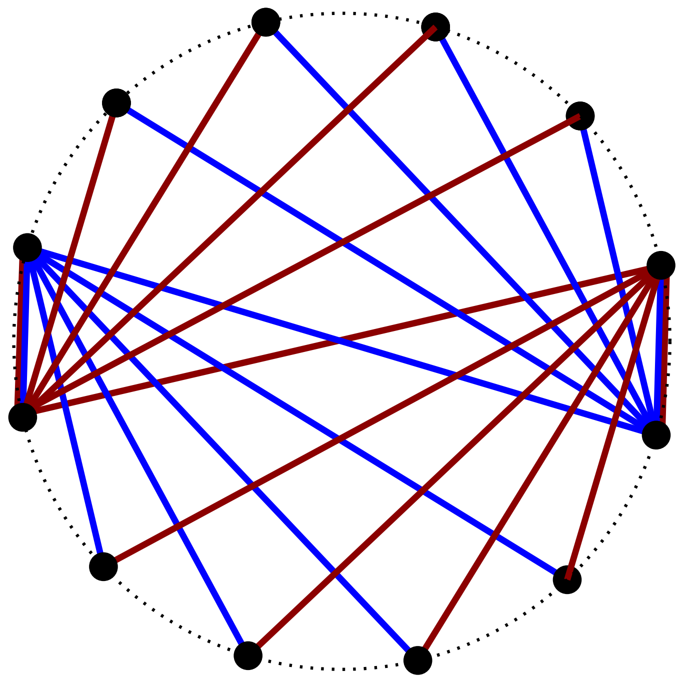
Spanning Trees — Simultaneous Operations

Upper and lower bounds for the diameter under *simultaneous* operations.

Operation	Simultaneous Upper Bound	Simultaneous Lower Bound
Exchange	1	1
Compatible Ex.	$O(\log n)$ [AAH02]	$\Omega(\frac{\log n}{\log \log n})$ [BRU+09]
Rotation	$O(\log n)$	$\Omega(\frac{\log n}{\log \log n})$
Empty-Tri. Rot.	$8n$	$\Omega(\log n)$
Edge Slide	$O(n^2)$ [AR07]	$\Omega(n)$

Convex Position		
Empty-Tri. Rot.	4	3
Edge Slide	$O(\log n)$	$\Omega(\log n)$

Spanning Trees — Exchange Operation



Lower bound construction:
It takes $\lfloor \frac{3n}{2} \rfloor - 5$ exchanges to transform T_1 to T_2.
[Hernando, Hurtado, Márquez, Mora, and Noy, 1999]

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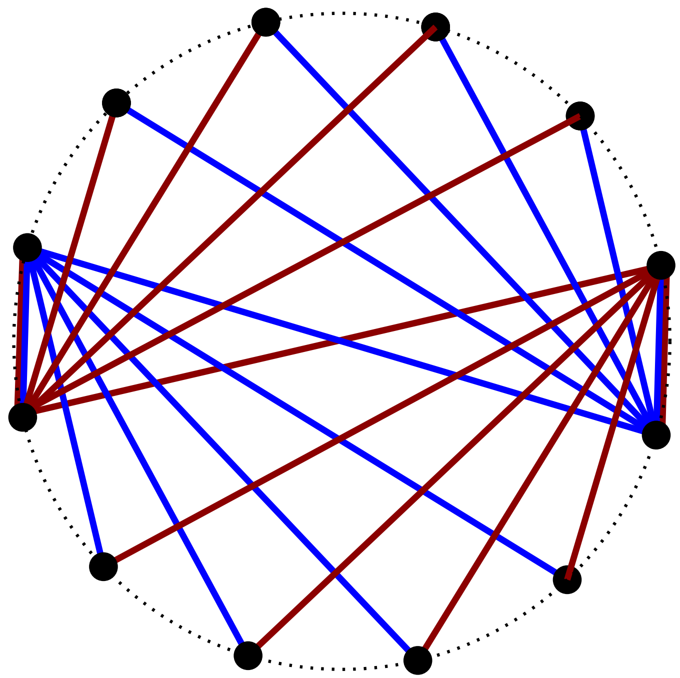

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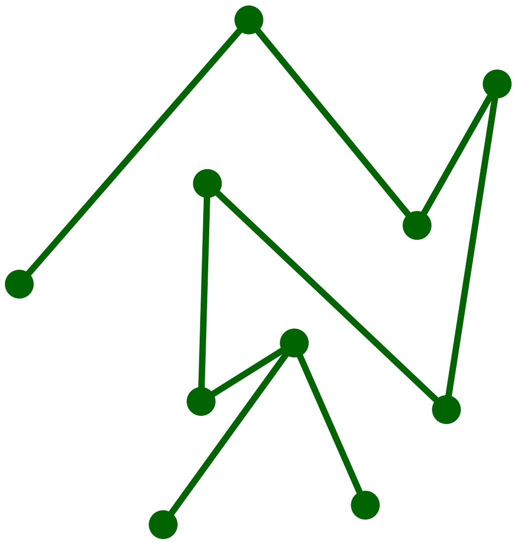
The same consturction gives a lower bound of $\lfloor \frac{3n}{2} \rfloor - 4$ for rotation operations.

Spanning Trees — Exchange Operation

$n-2$ exchanges can transform any plane graph into a star centered at the convex hull.
$\Rightarrow$ Diameter $\leq 2n-4$
[Avis & Fukuda, 1996]



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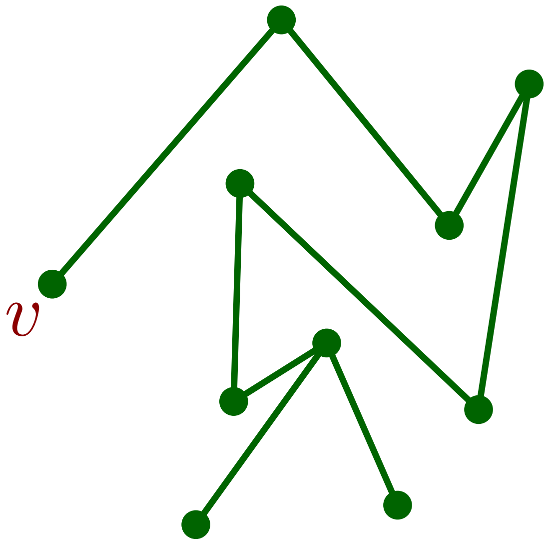

Let v be a vertex on the convex hull.
While T is not a star centered at v,

- v sees an entire edge ab.
- Rotate ab to av or bv.

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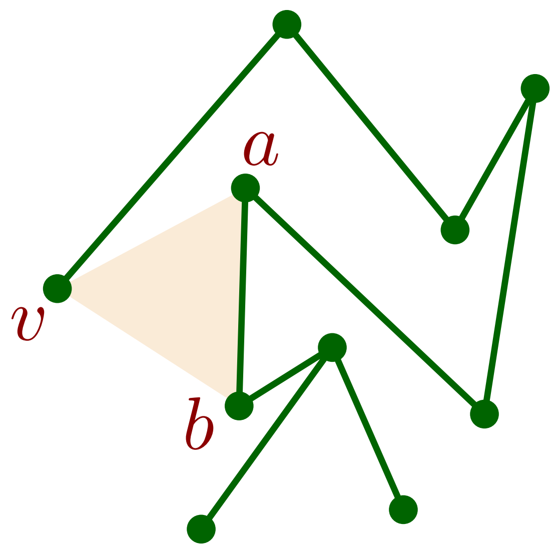

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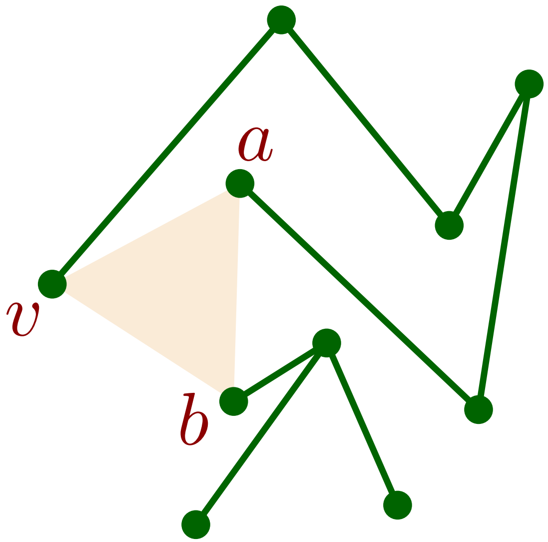

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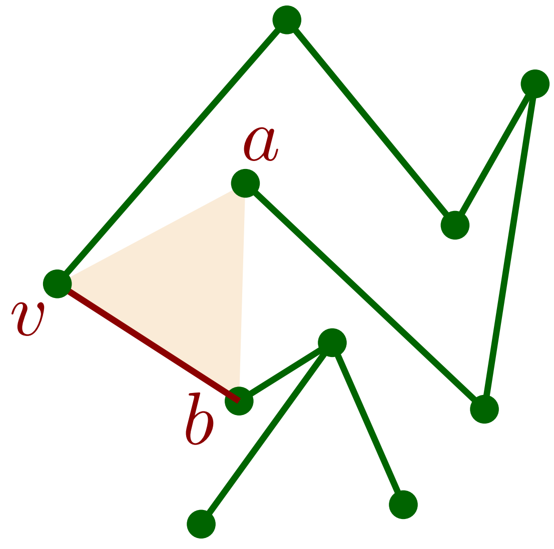

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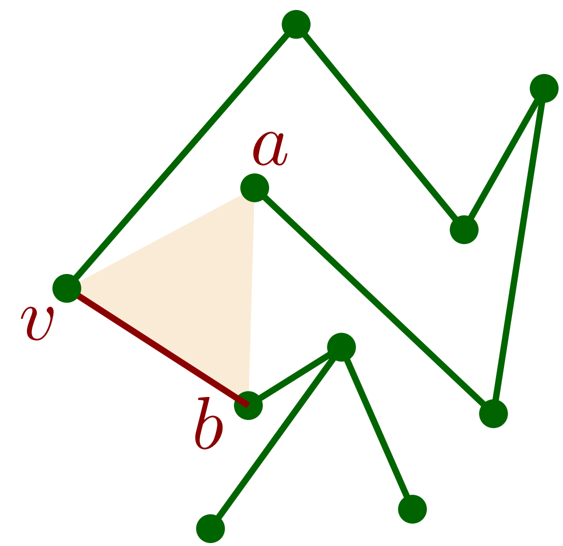

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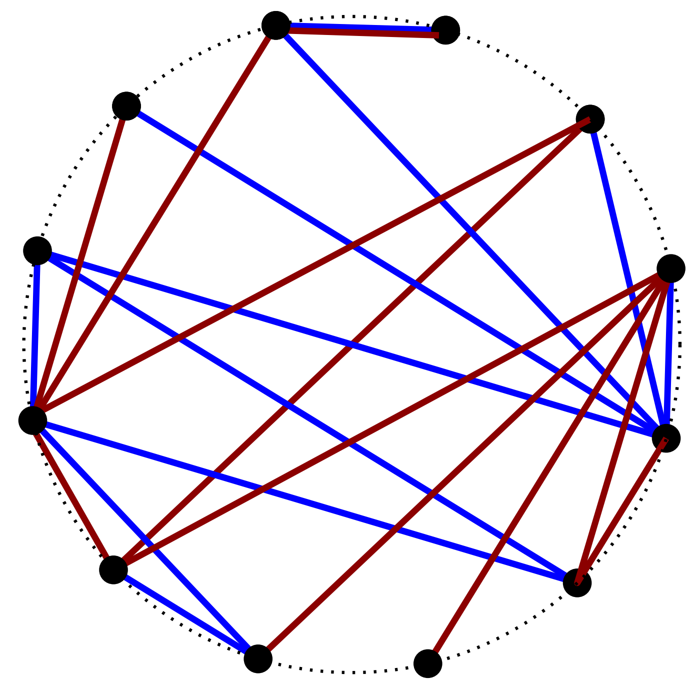
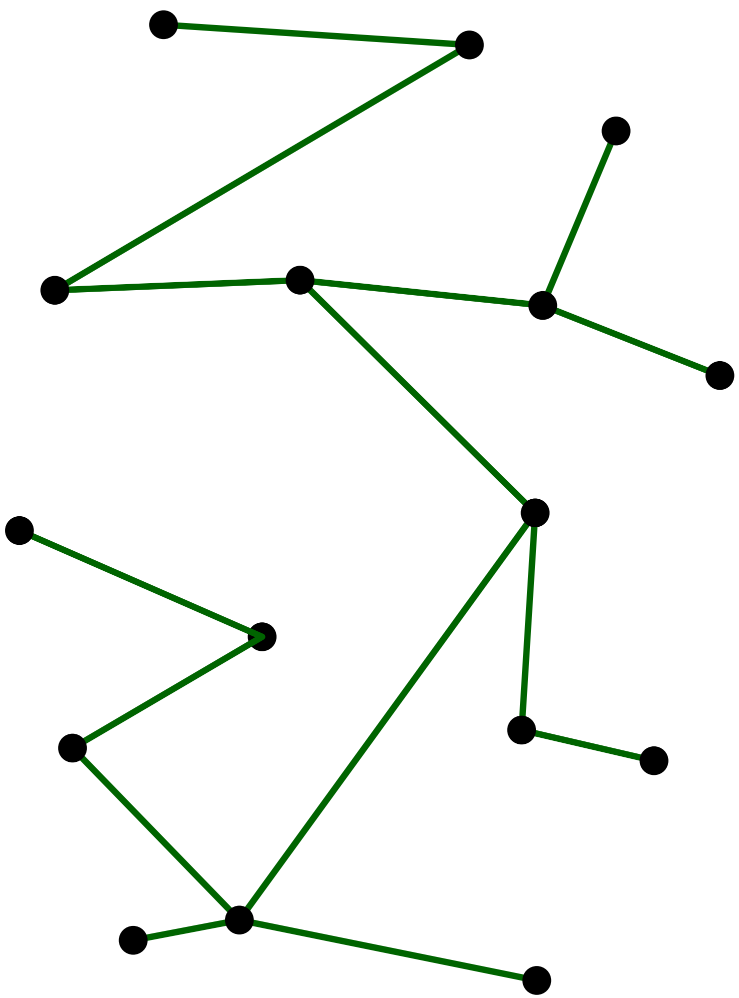


For $n \geq 3$ points in convex position: diameter $\leq \frac{23n}{12} - 5$.
[Lonner & T., 2018]

Spanning Trees — Empty-Triangle Rotation



At most $3n$ empty-triangle rotations can remove all but one edges between the two halves.
$f(n) \leq 3n + 2f(n/2)$
$\Rightarrow$ Diameter is $O(n \log n)$

Let ℓ be a halving line.
Triangulate T.
For every triangle Δ along ℓ (in stabbing order),

- If the first edge of Δ crossed by ℓ is in T, then replace it with another edge of Δ.

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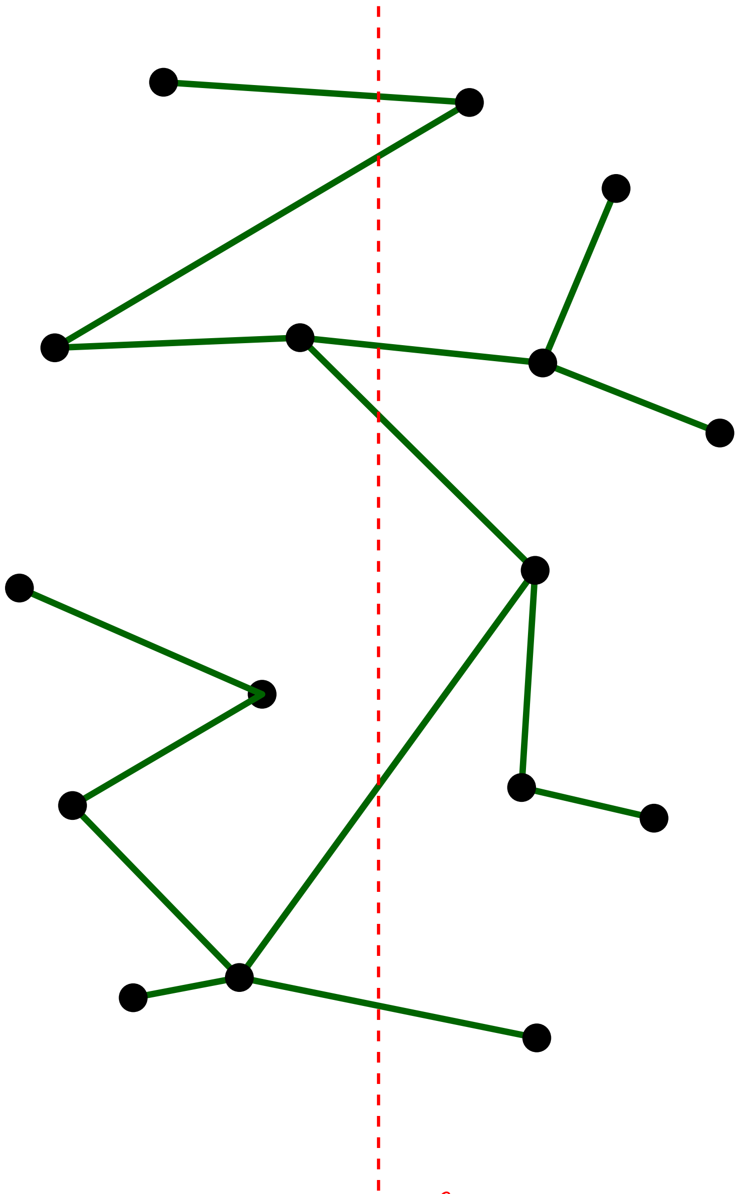

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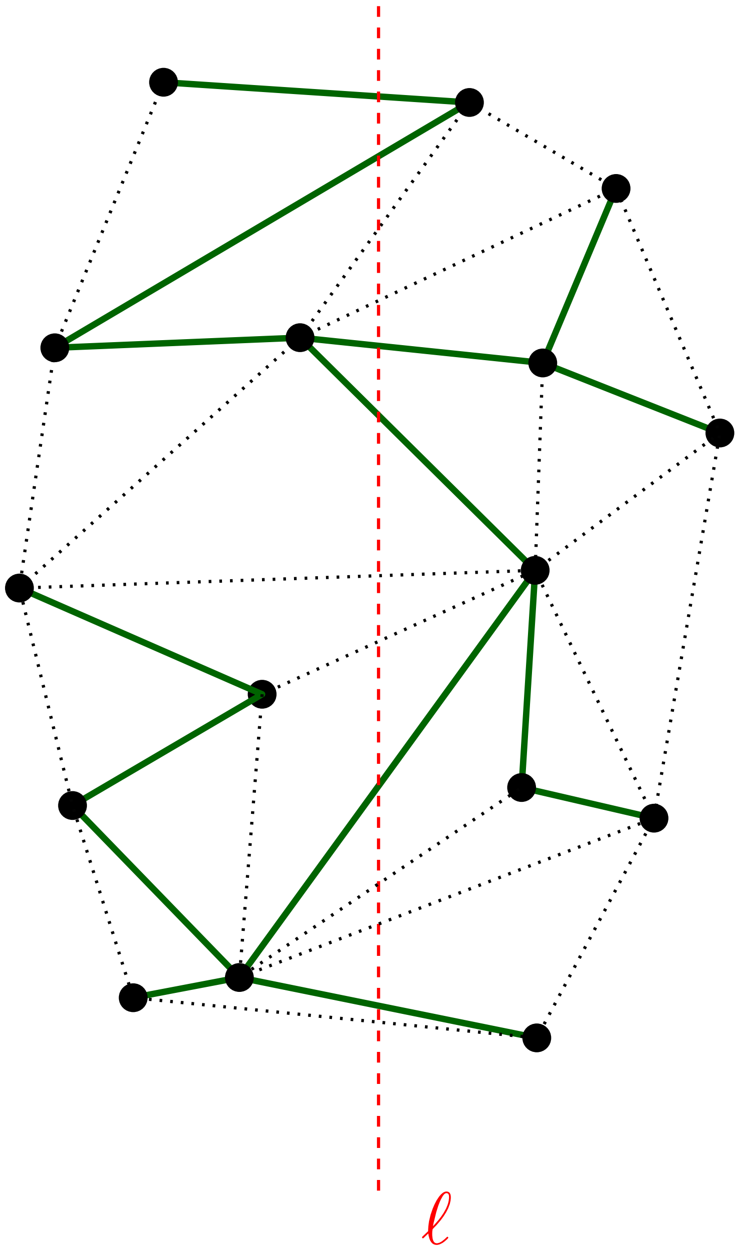

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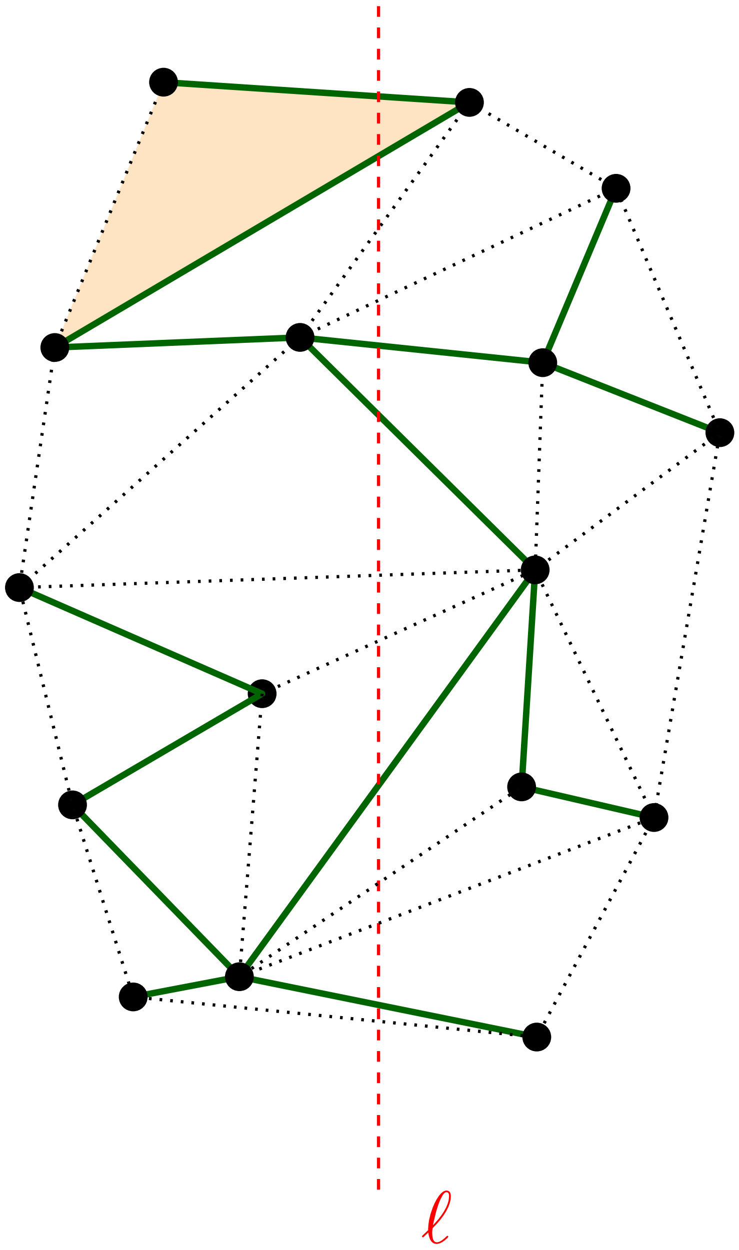

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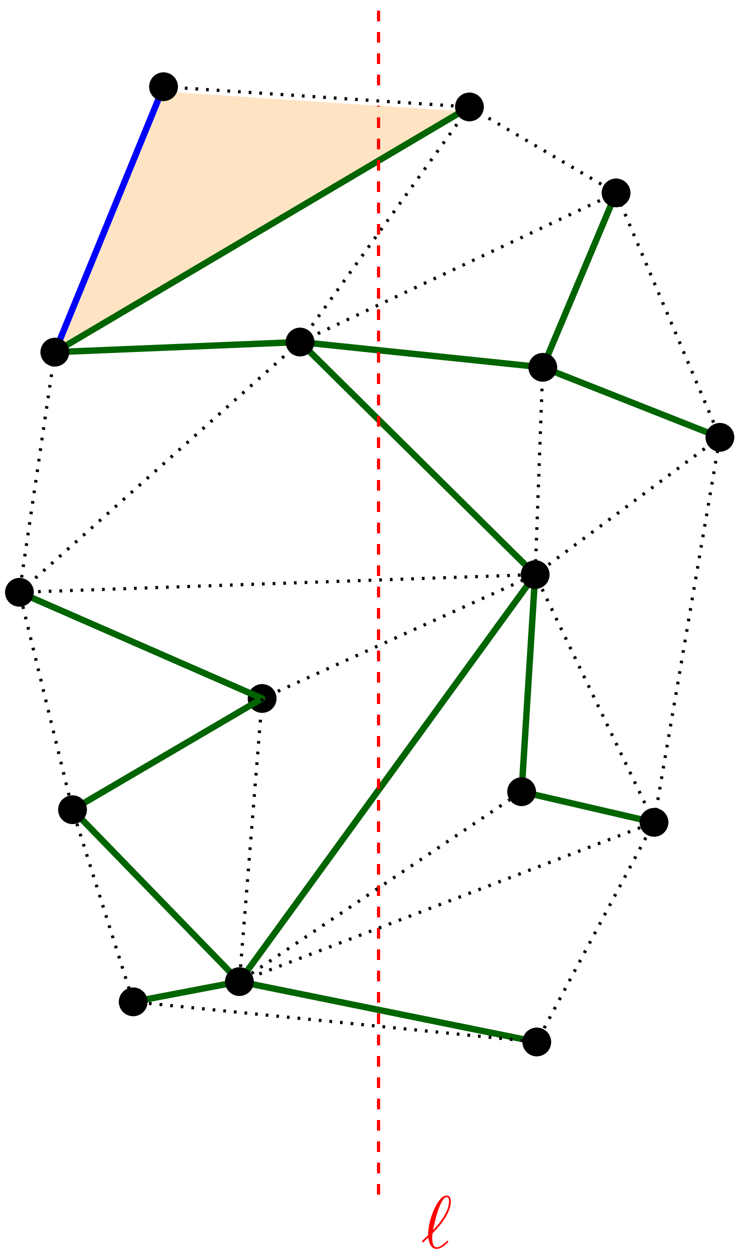

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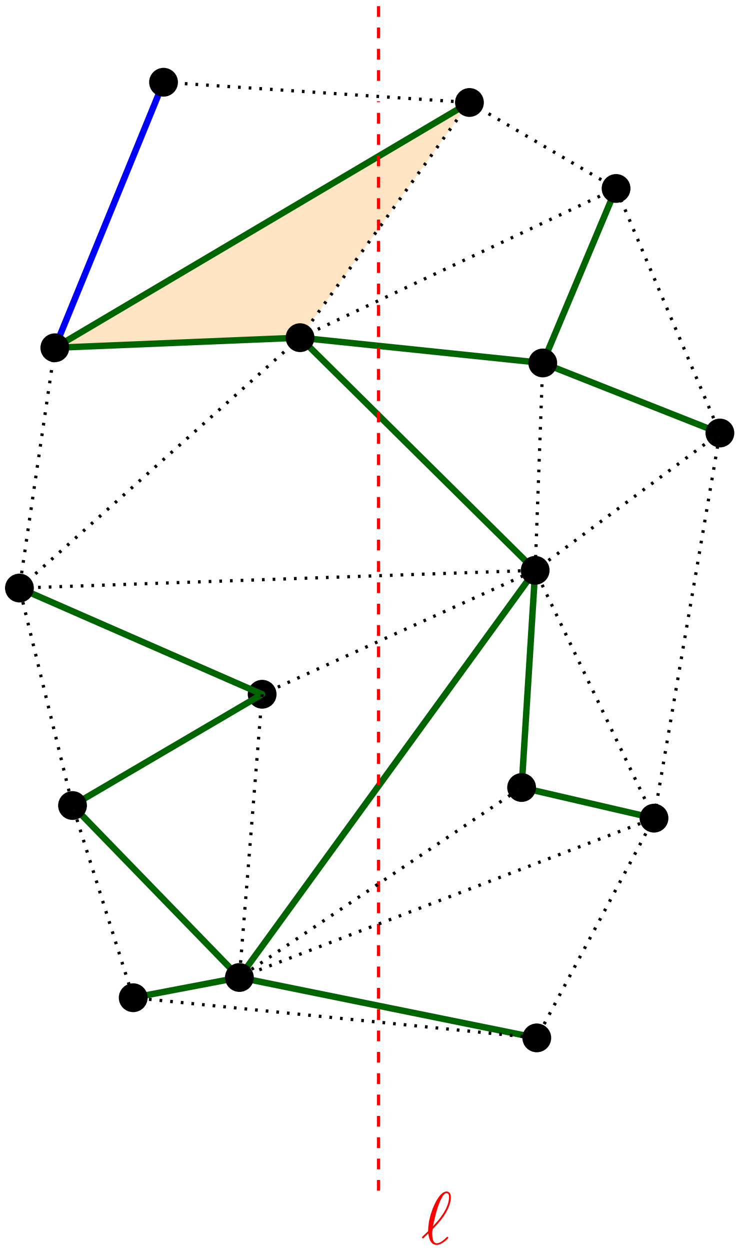

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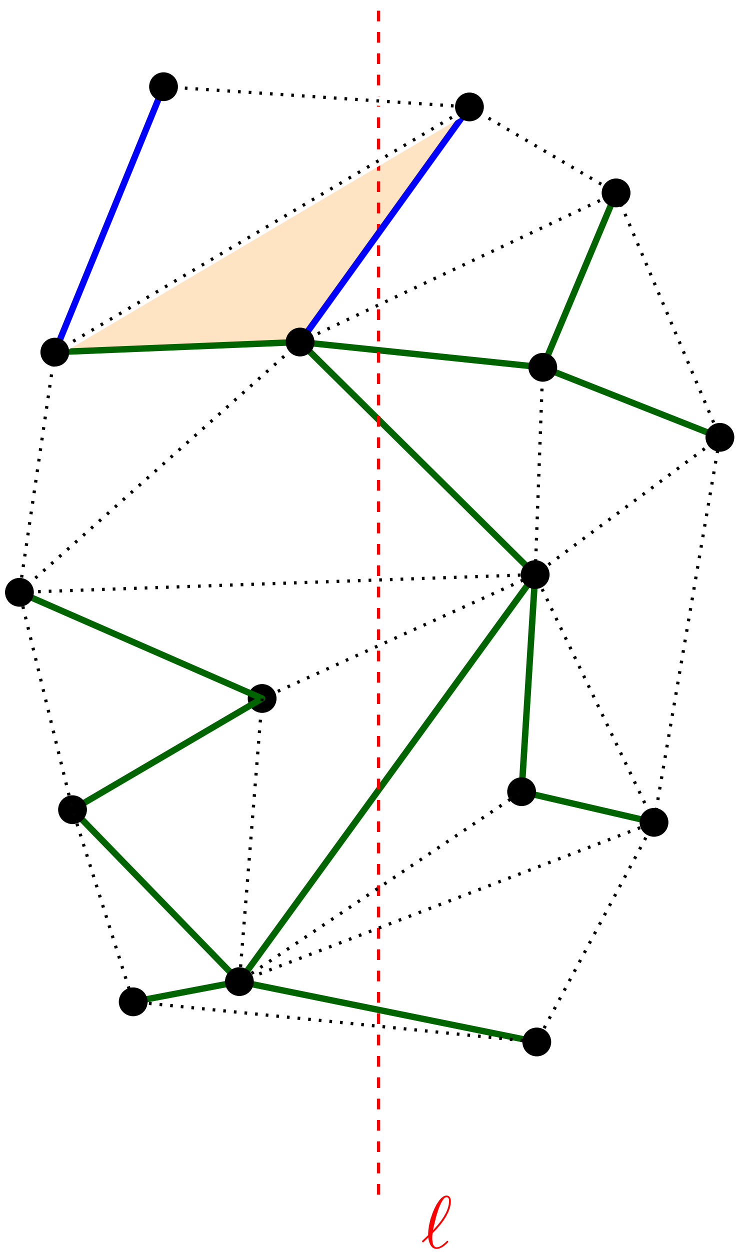

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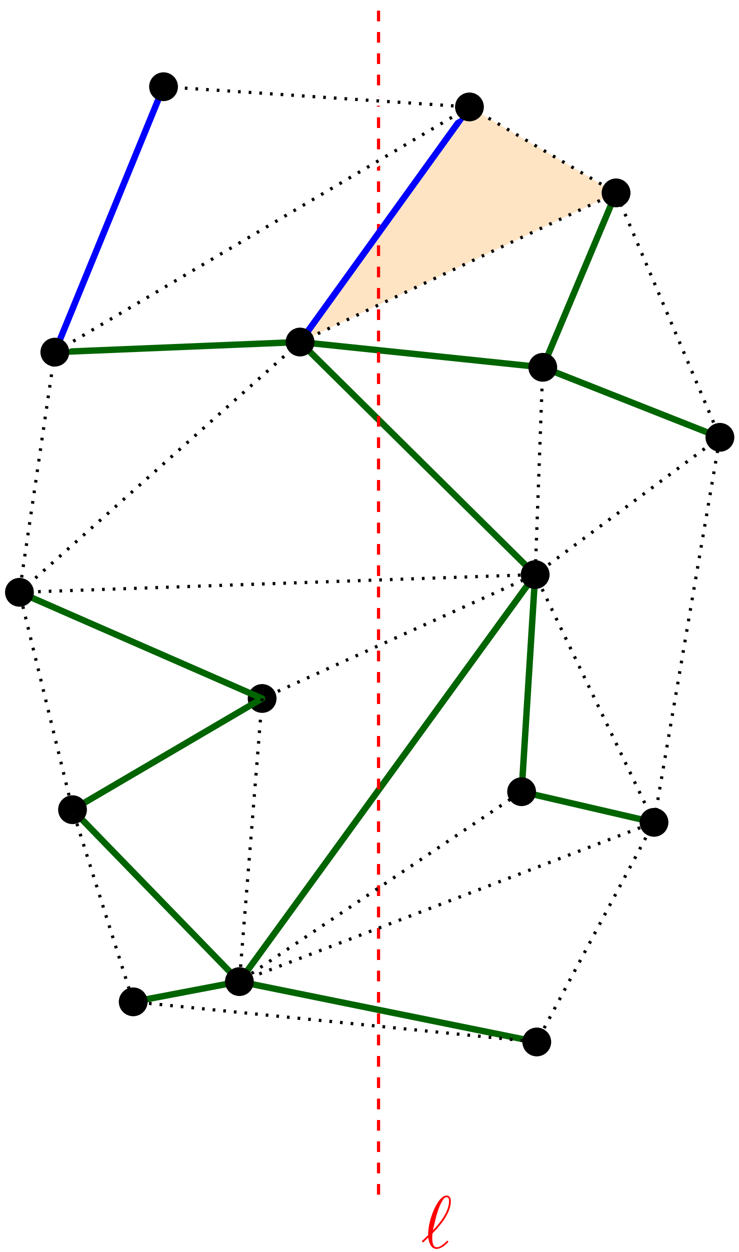

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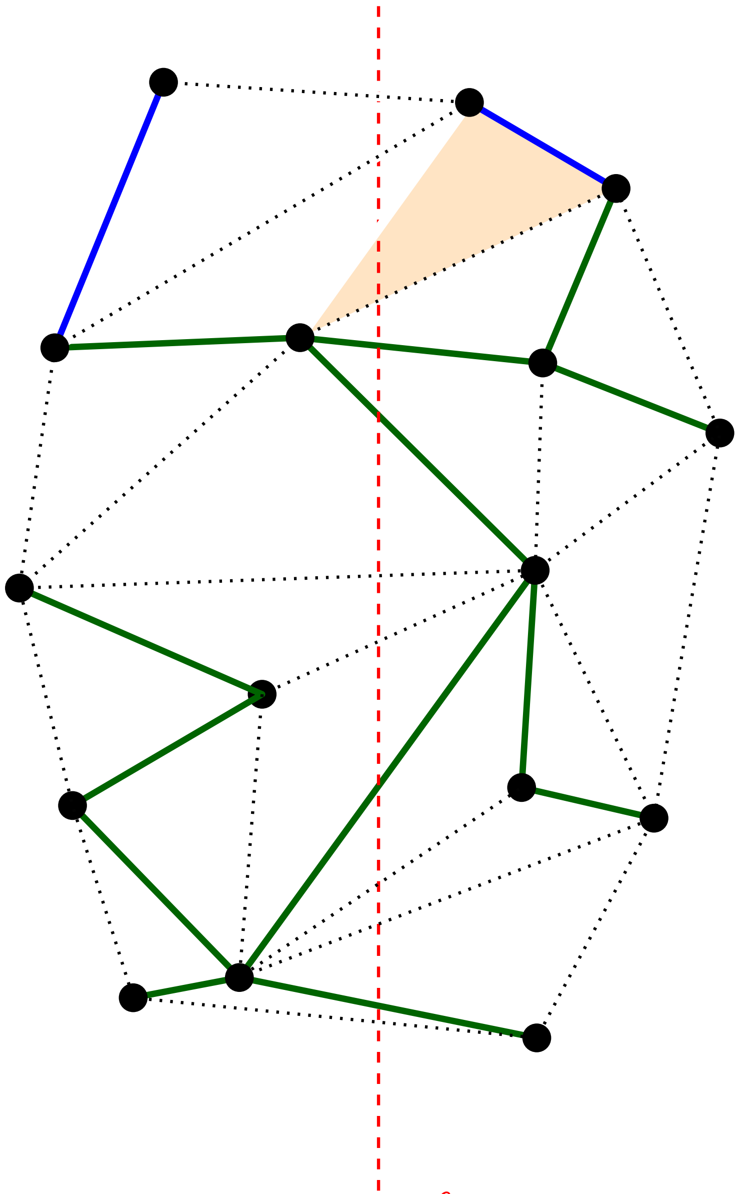

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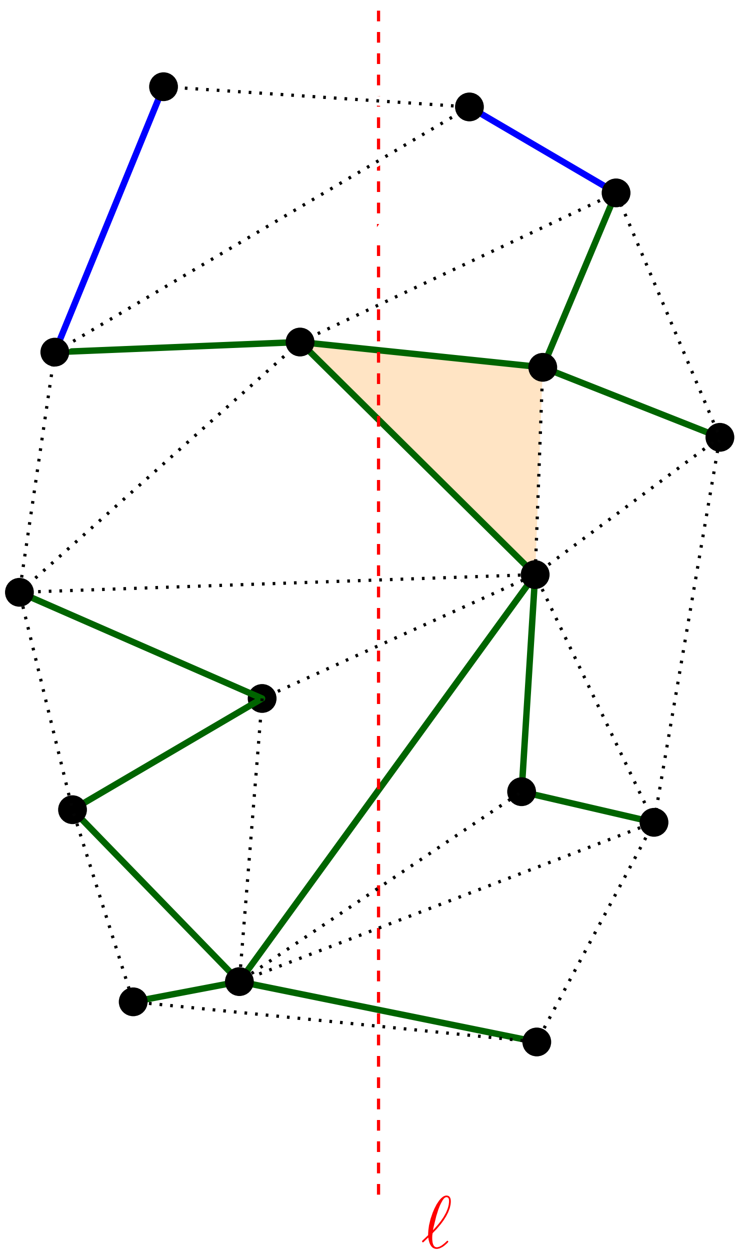

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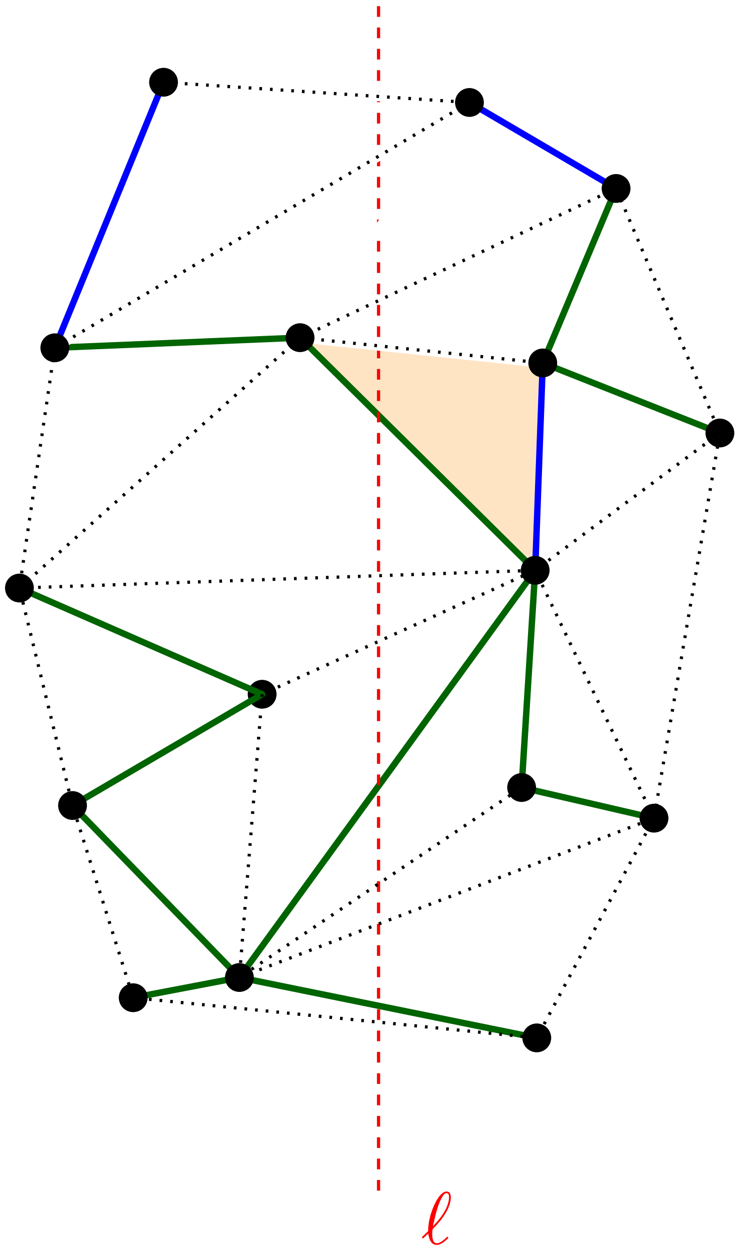

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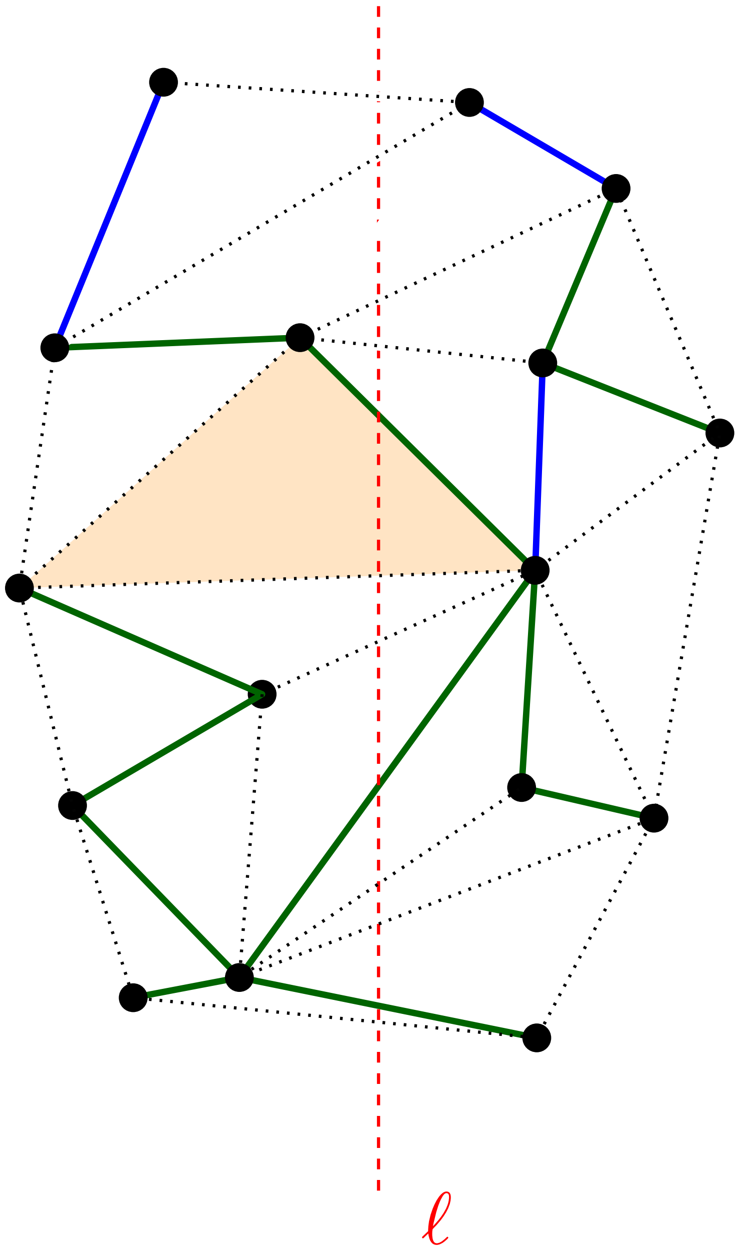

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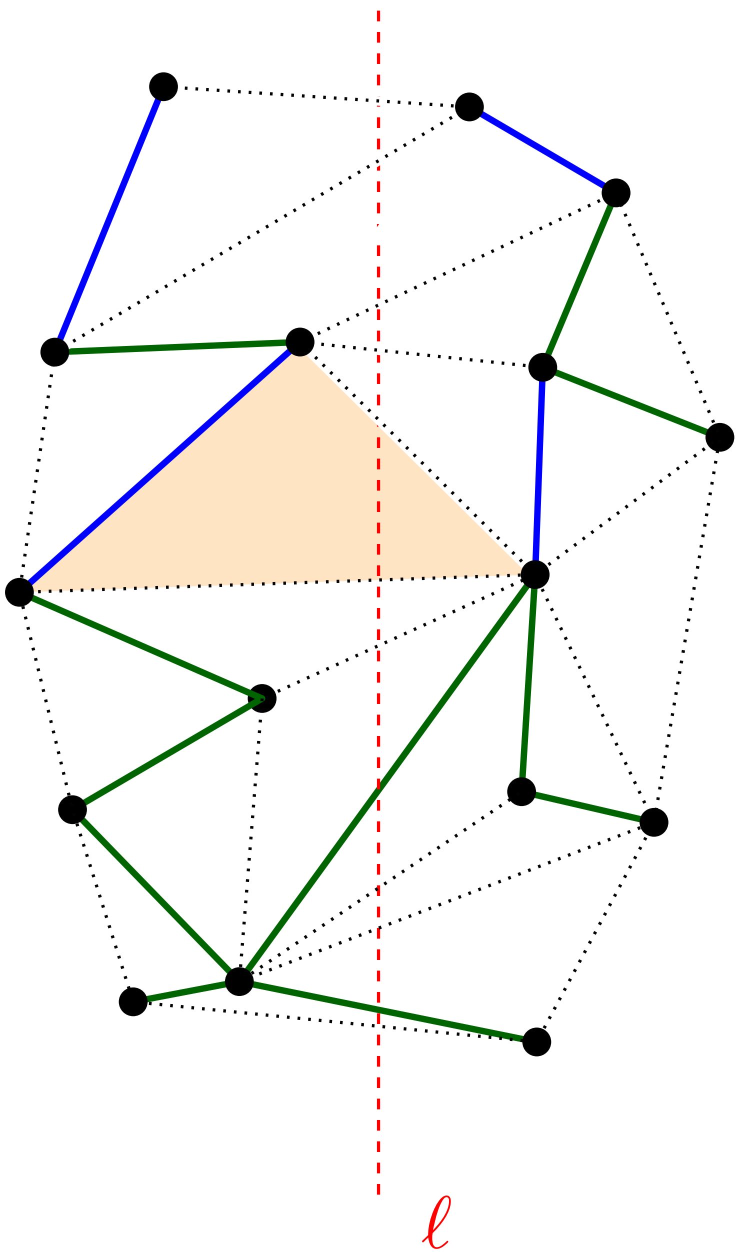

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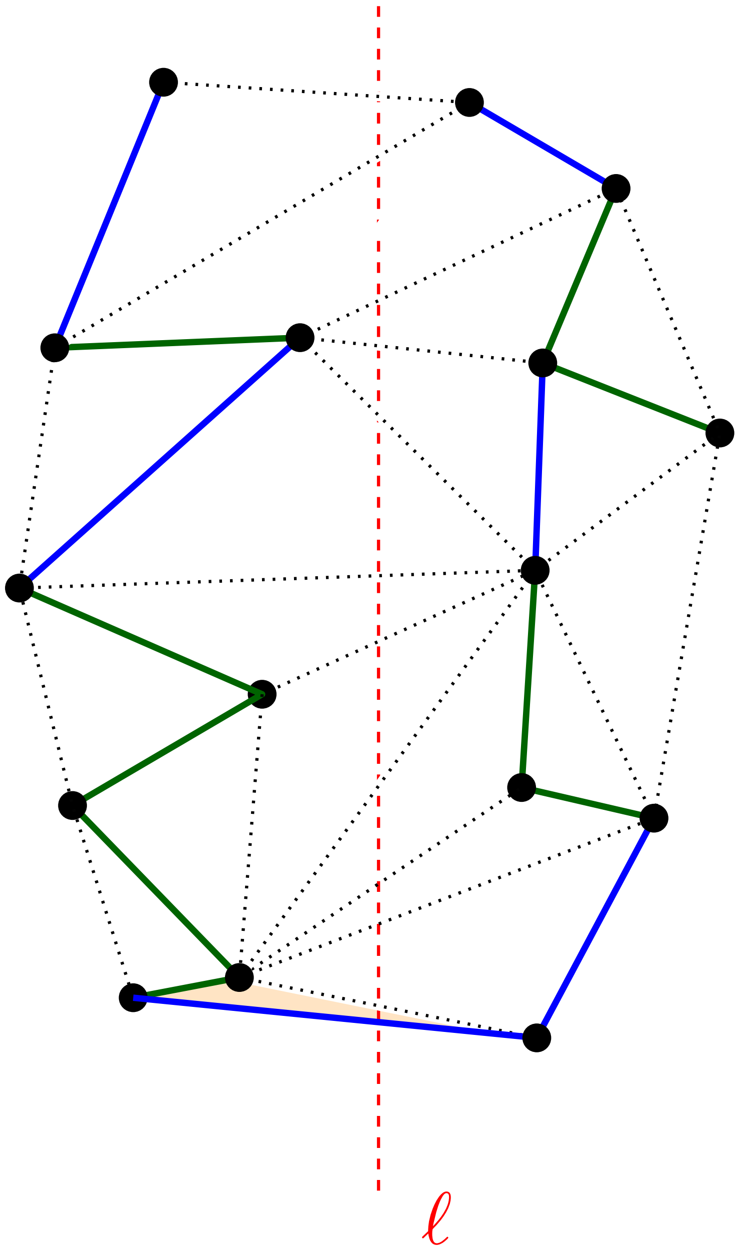

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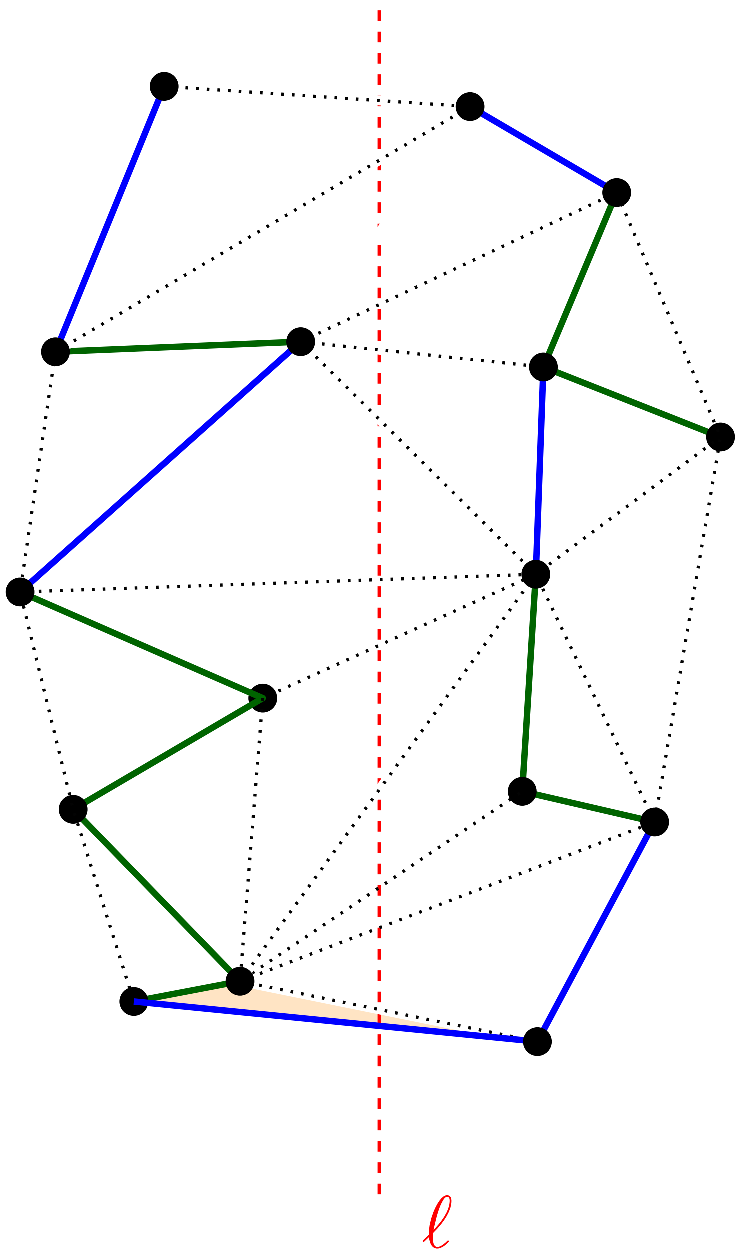
Simultaneous Empty-Triangle Rotation



At most $3n$ empty triangle rotations can remove all but one edges between the two halves.
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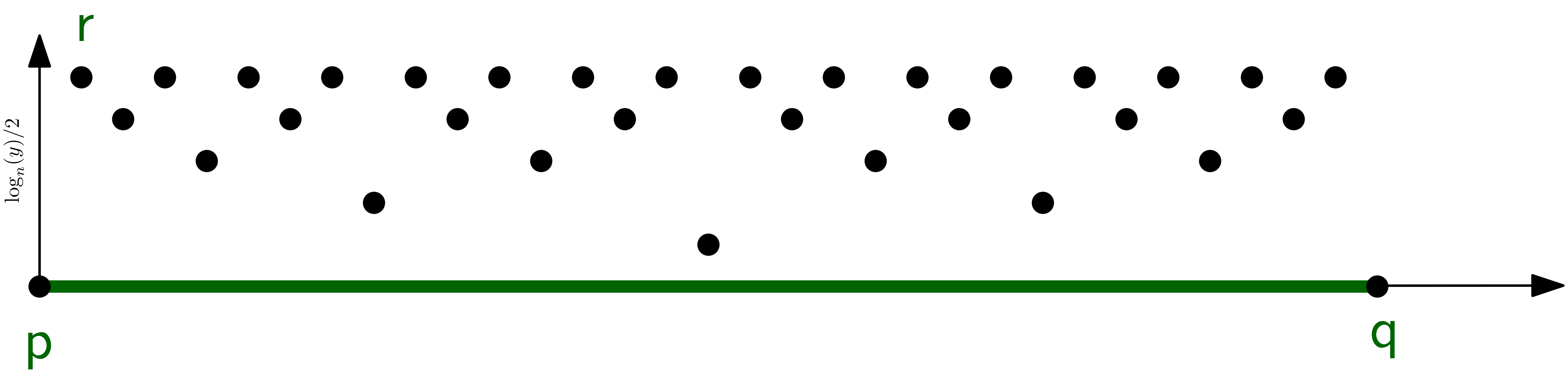
Simultaneous Empty Triangle Rotations

$\Omega(\log n)$ simultaneous empty-triangle rotations are sometimes necessary:

Tree T_1 contains a horizontal edge pq.

Tree T_2 is a star centered at r.



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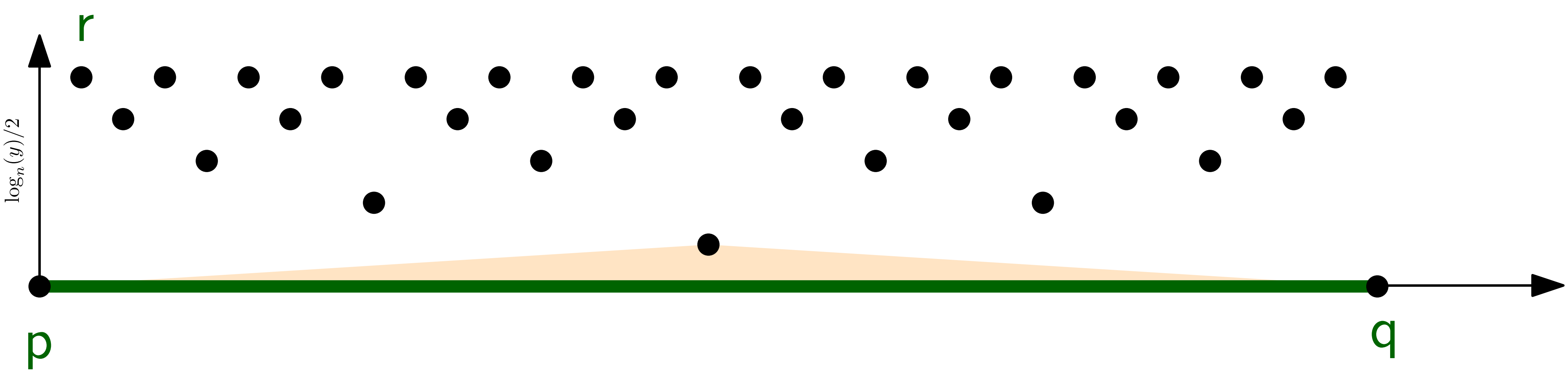

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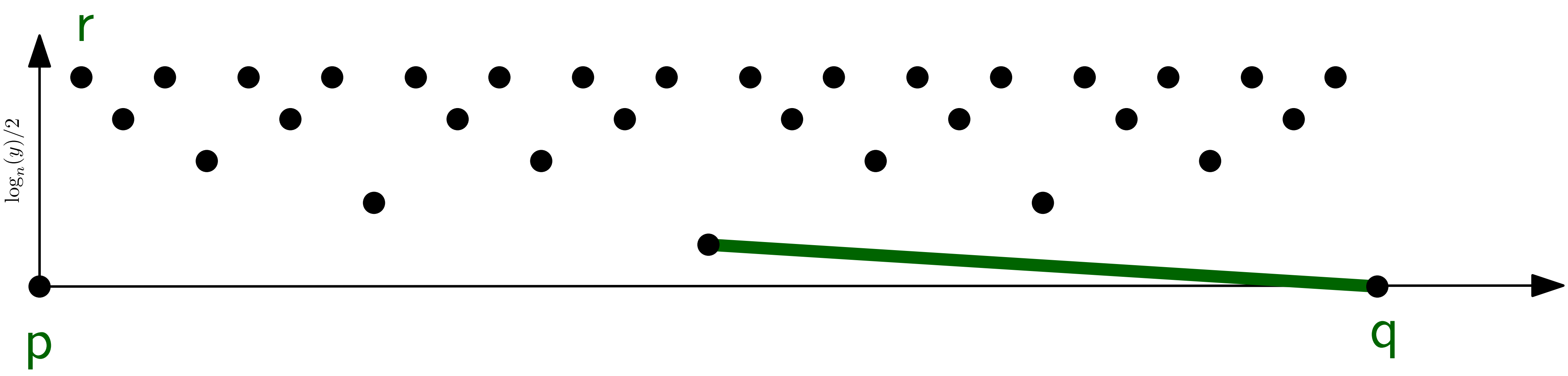

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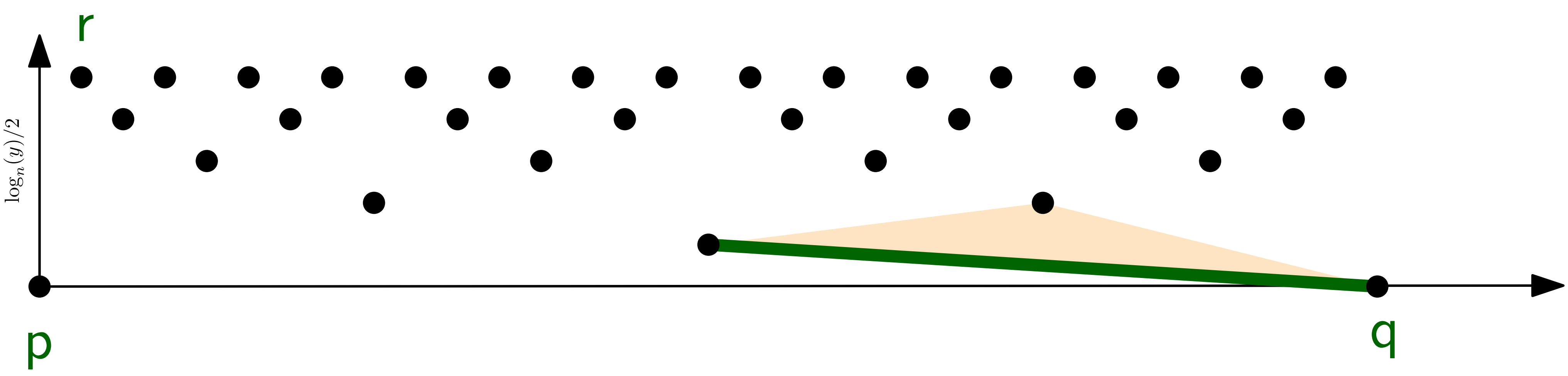

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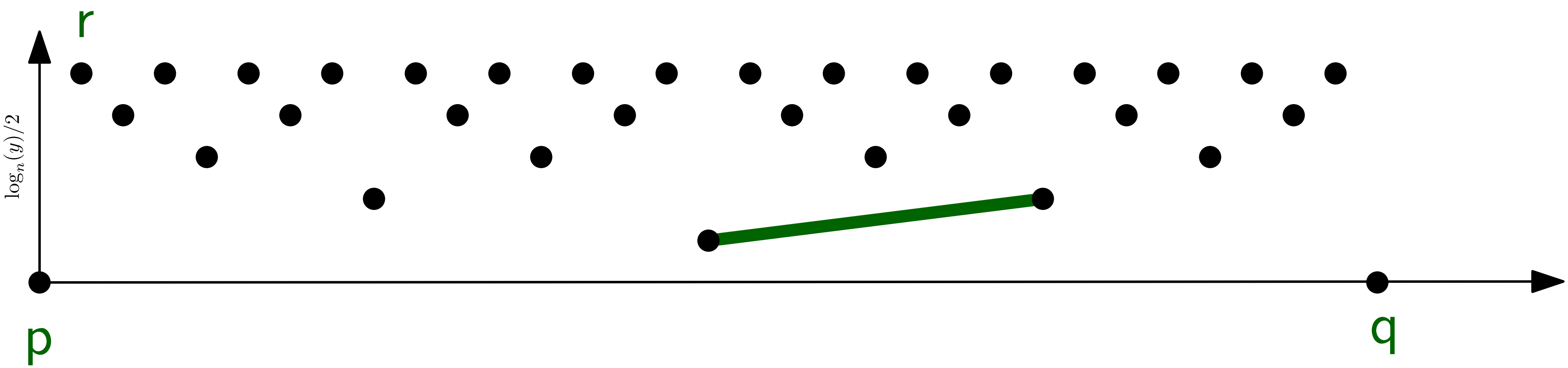

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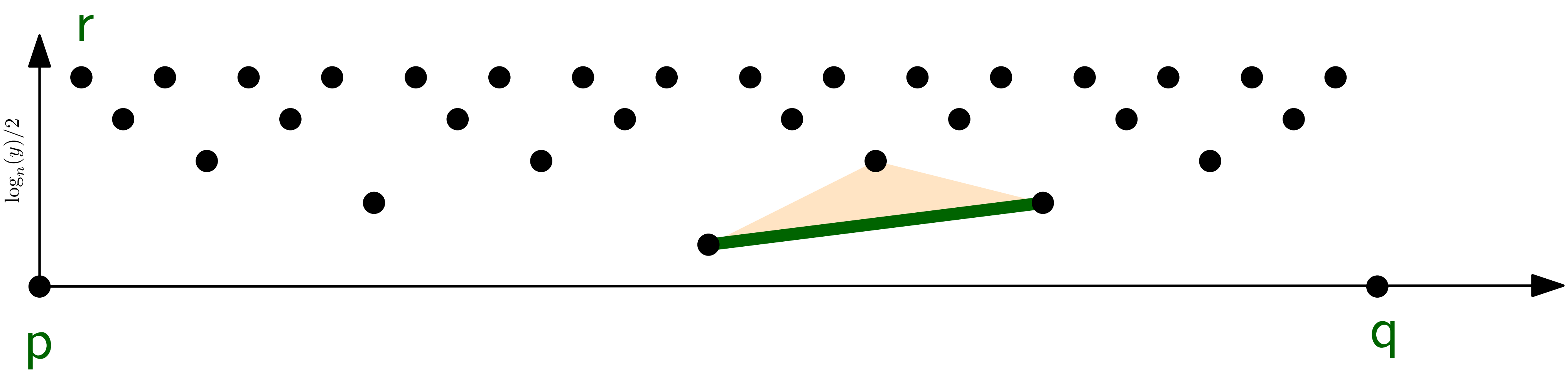

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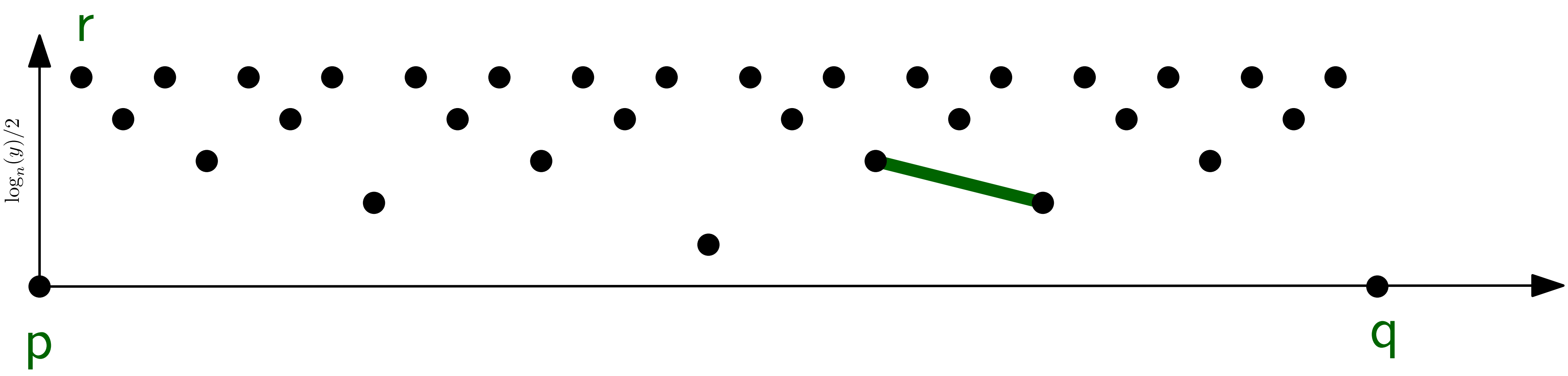

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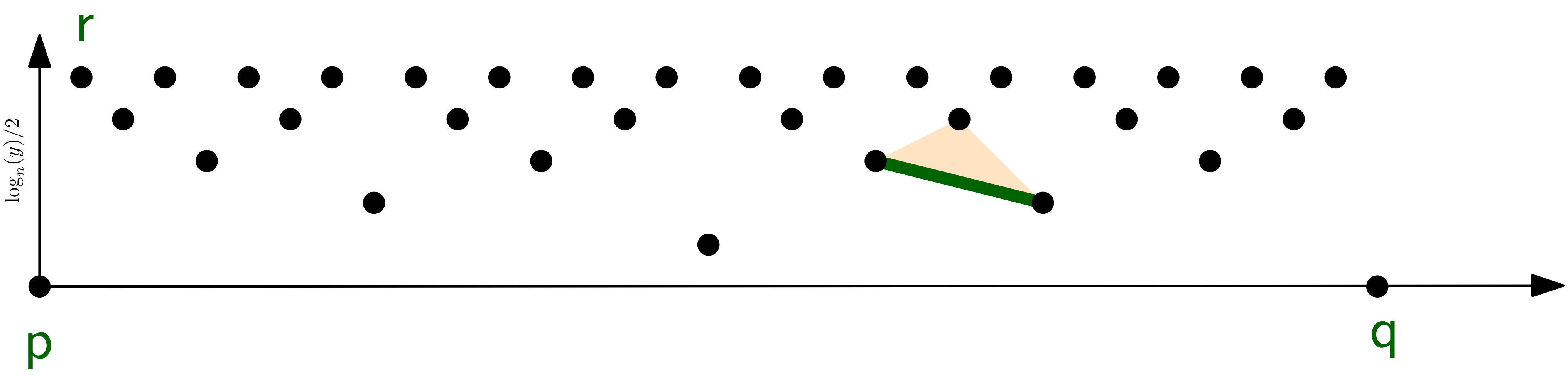

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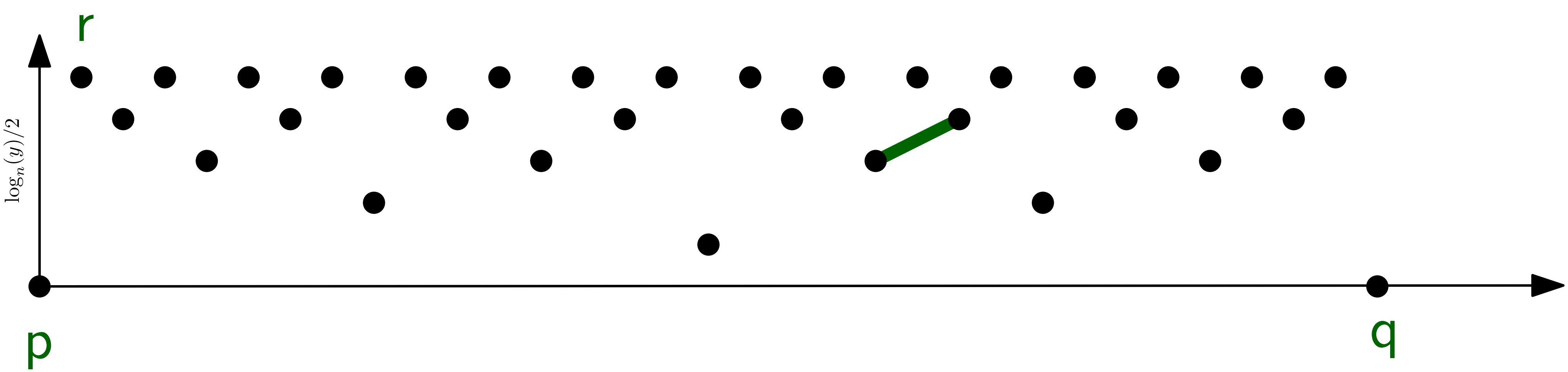

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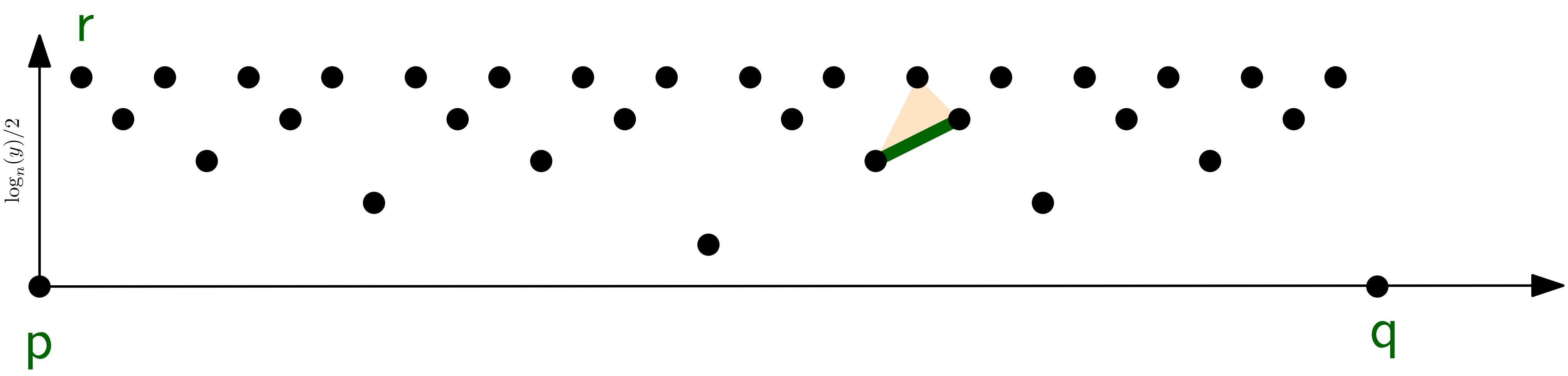

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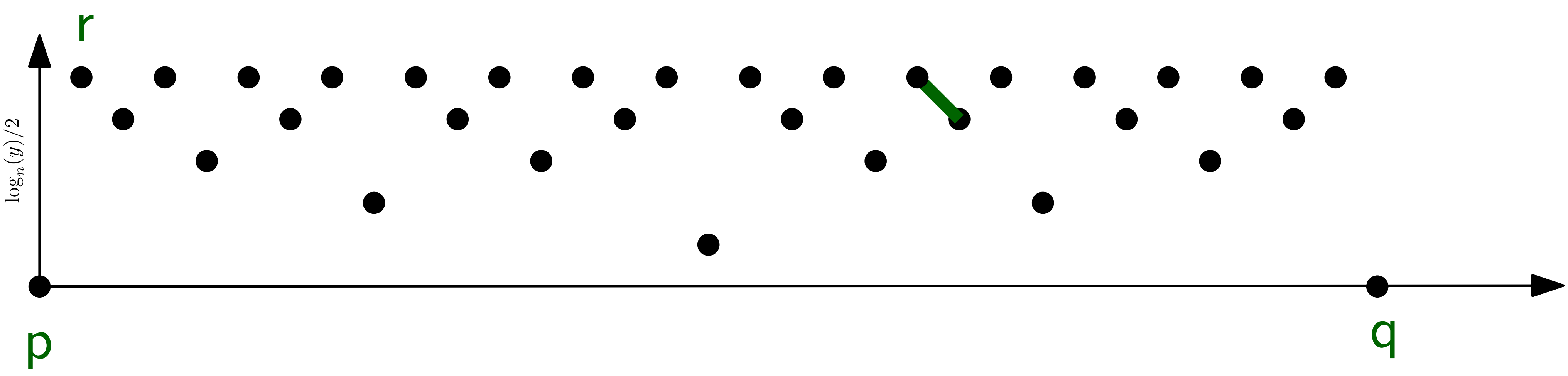

Simultaneous Empty Triangle Rotations

$\Omega(\log n)$ simultaneous empty-triangle rotations are sometimes necessary:

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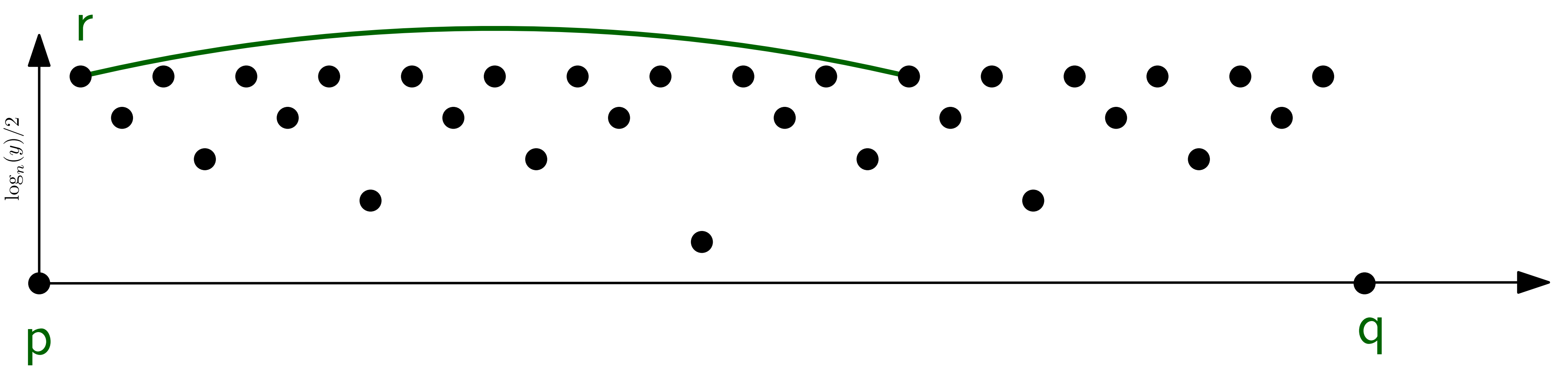

Spanning Trees — Simultaneous Rotations

$O(\log n)$ simultaneous rotations can transform any plane graph into a star centered at the convex hull.

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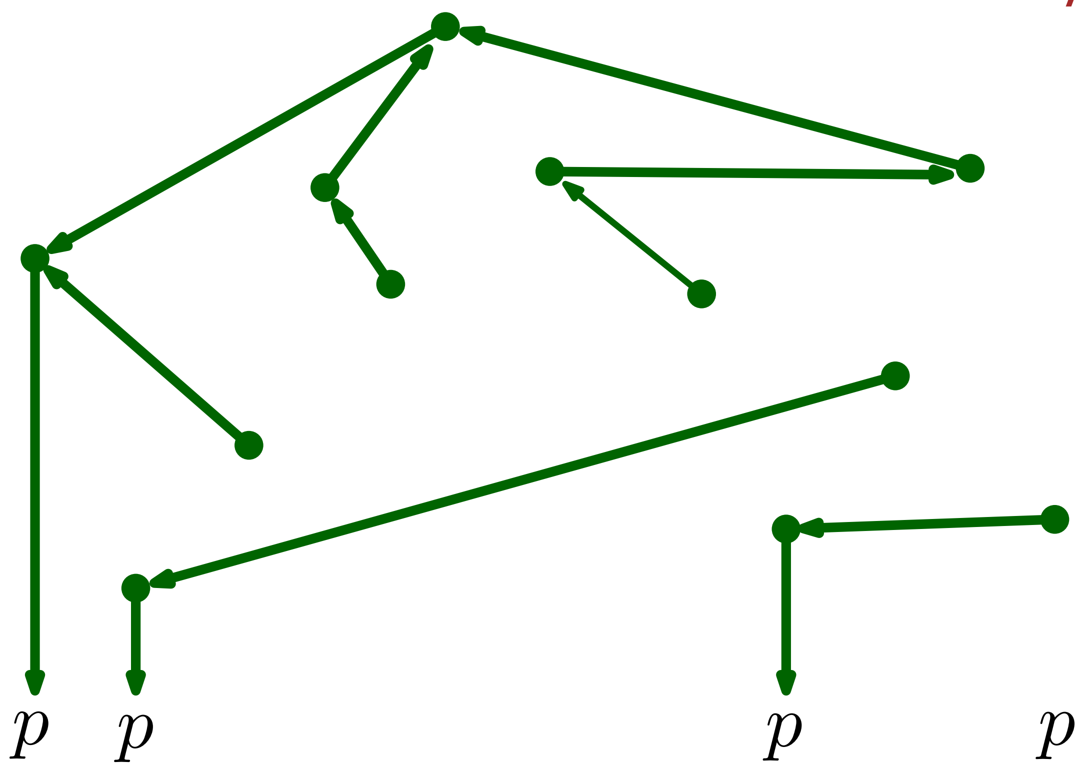

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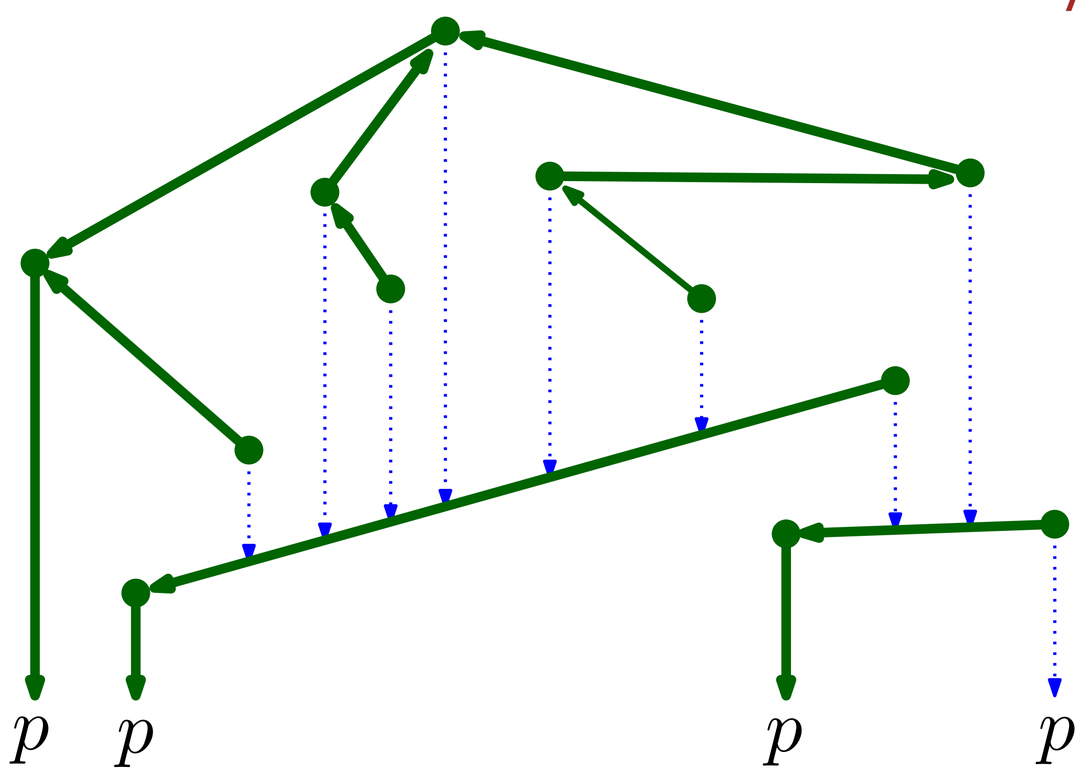

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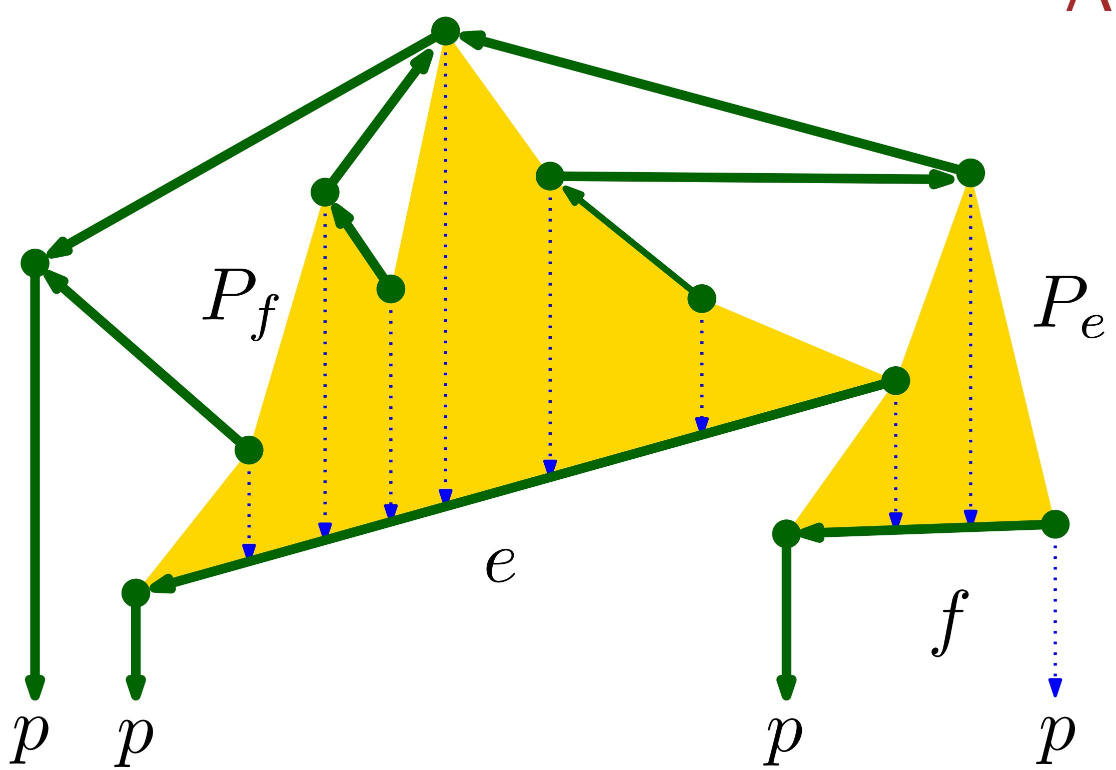

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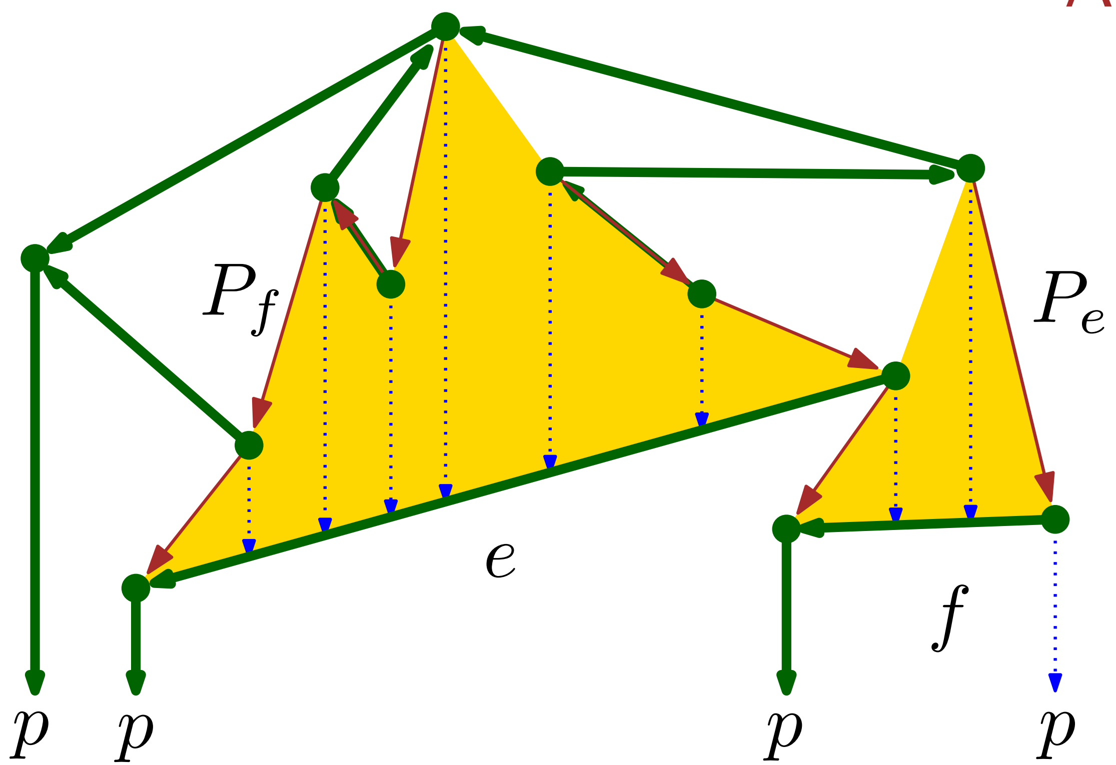

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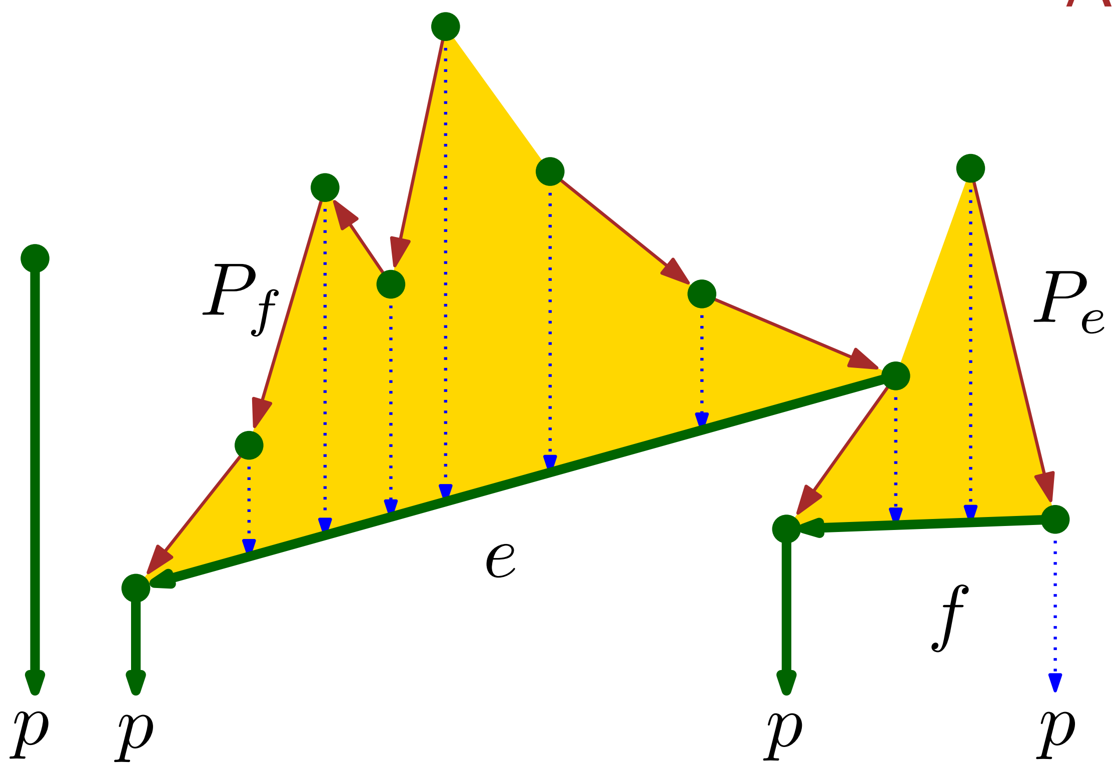

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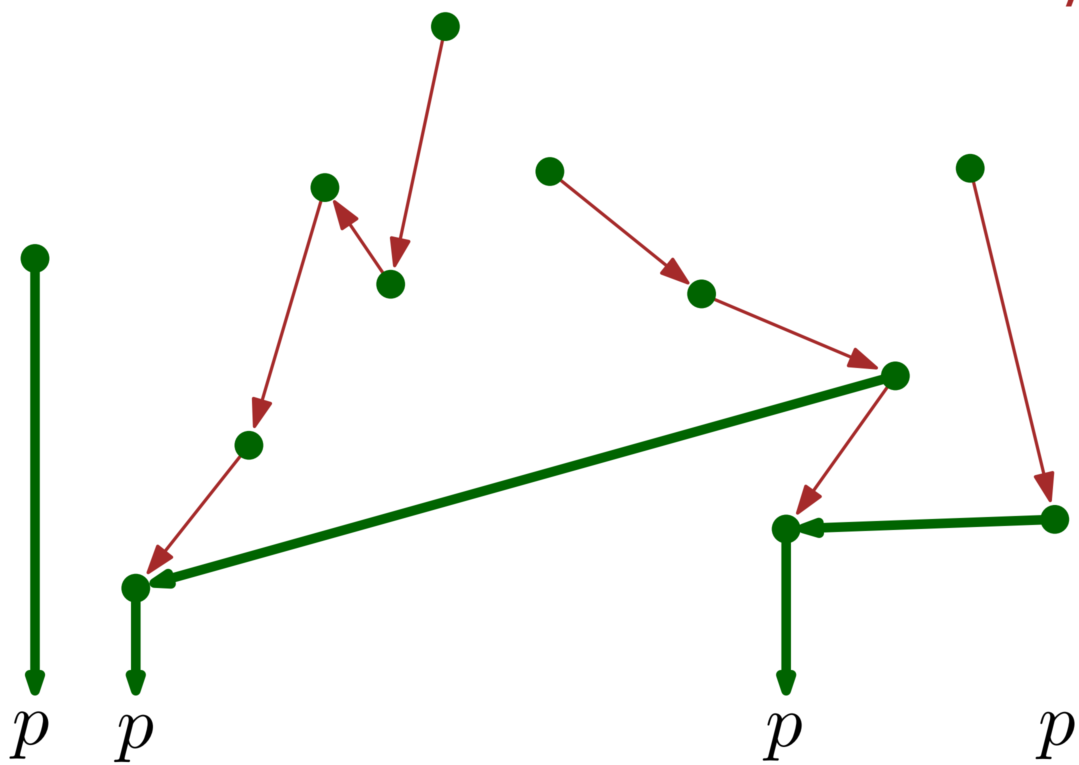

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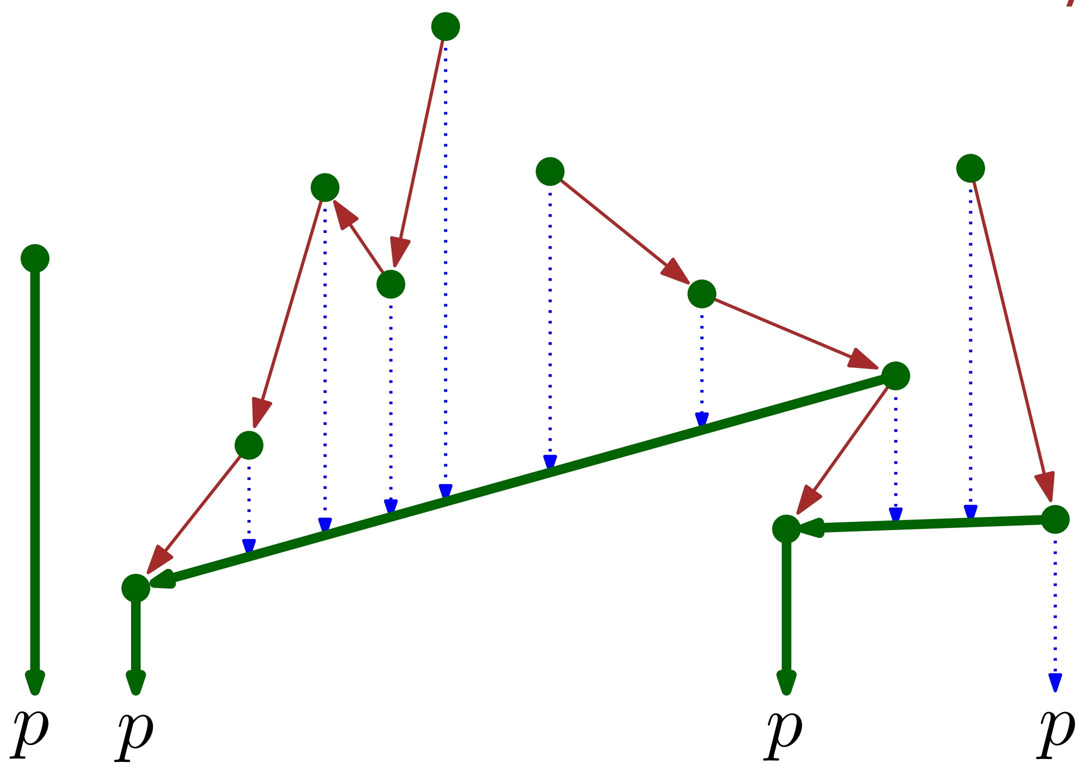
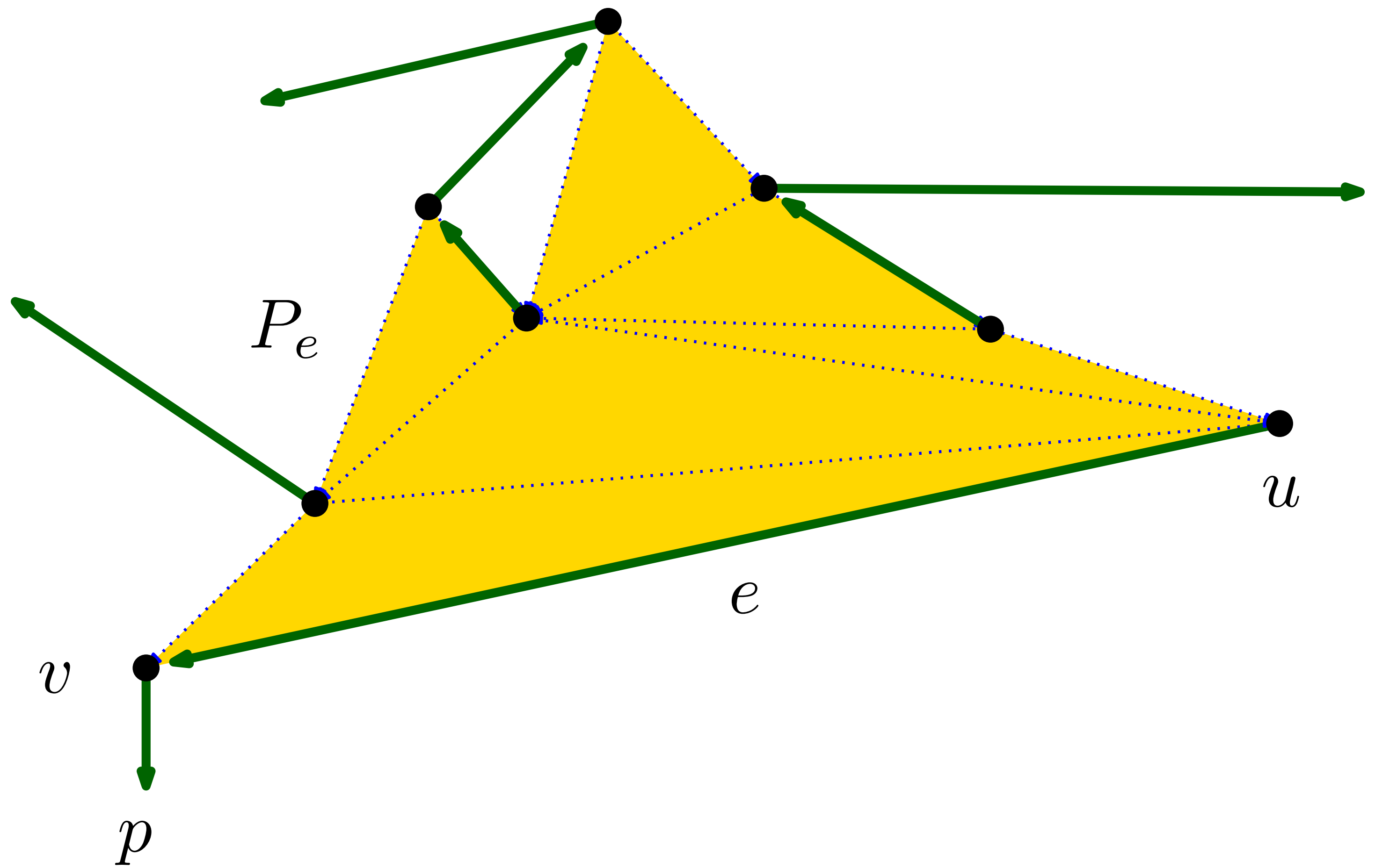


`starify`(p) maintains a plane spanning tree. The sum of "discretre" horizontal extents all edges decreases by a factor of $\frac{1}{2}$.

$\Rightarrow$ Algo. terminates after $O(\log n)$ moves.

Spanning Trees — Simultaneous Rotations

Each iteration of `starify`(p) can be implemented in at most 4 **simultaneous rotations**.



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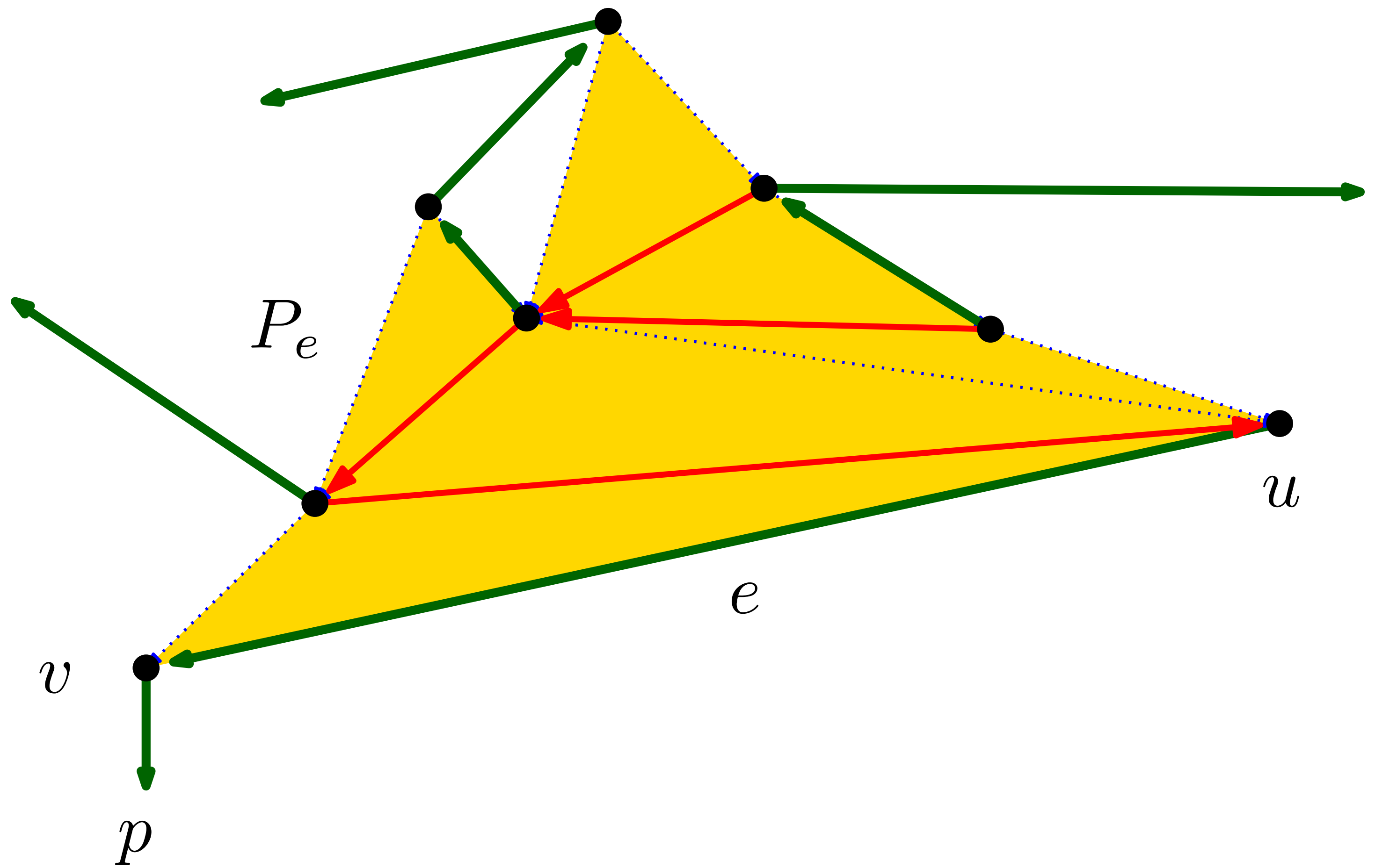

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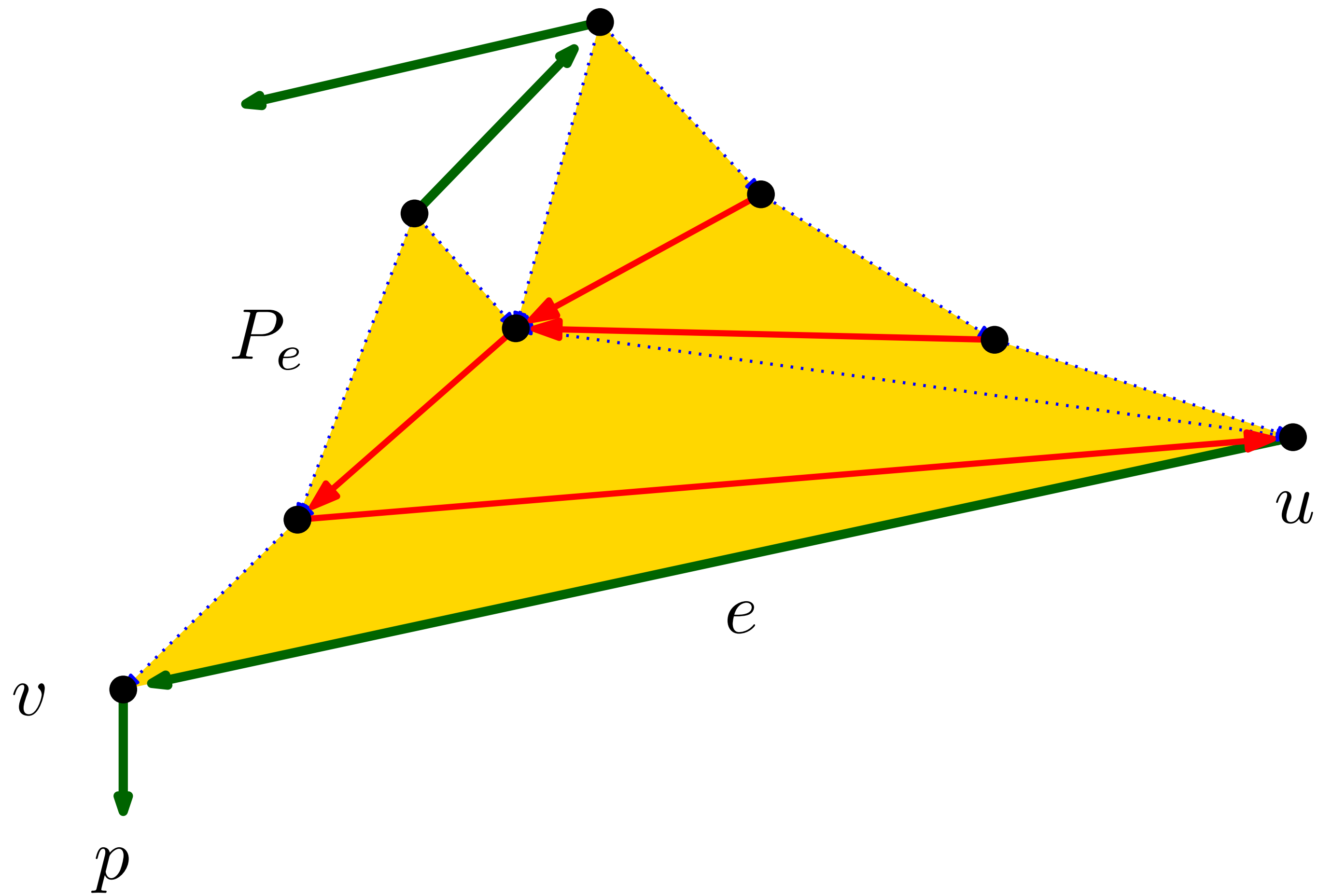

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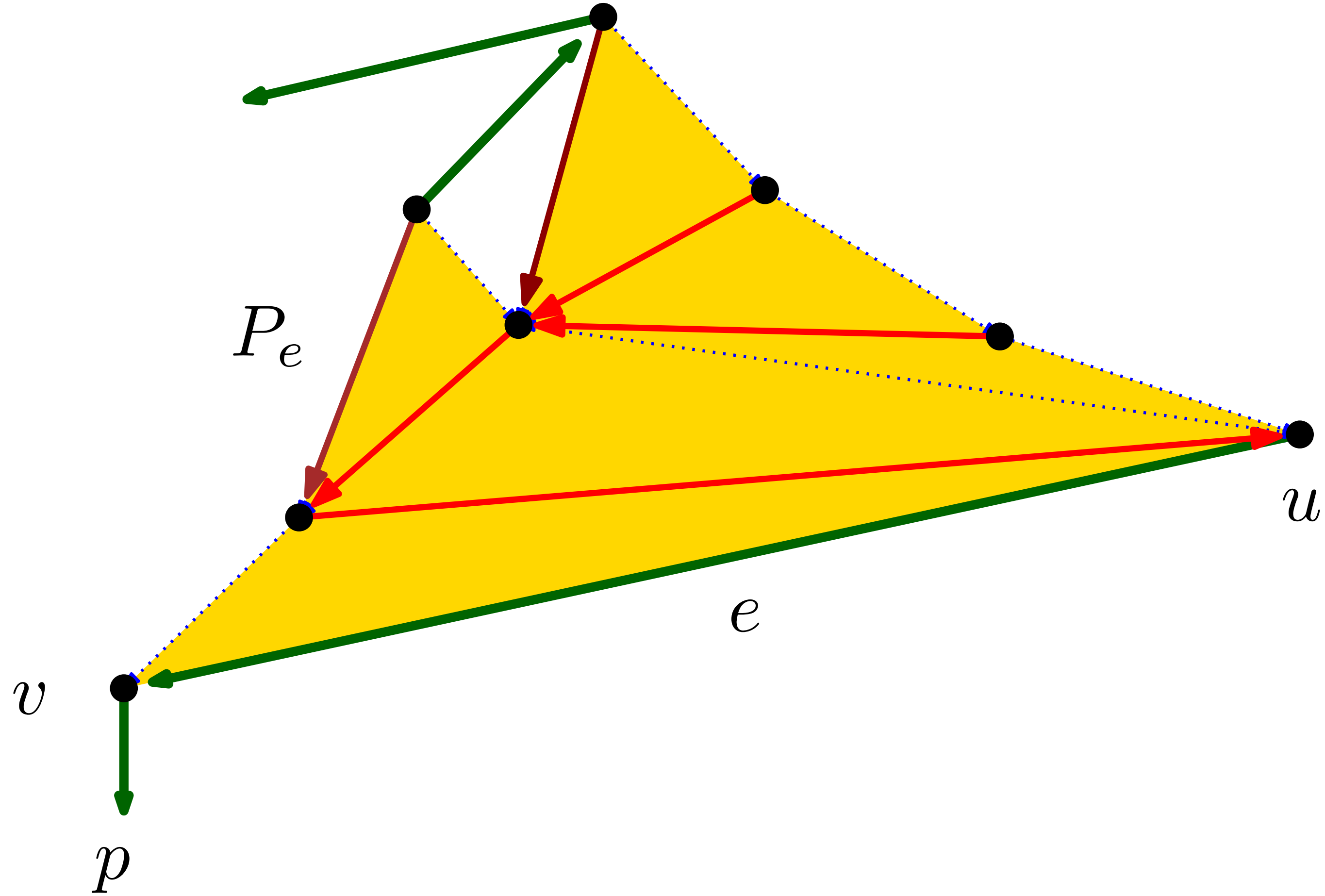

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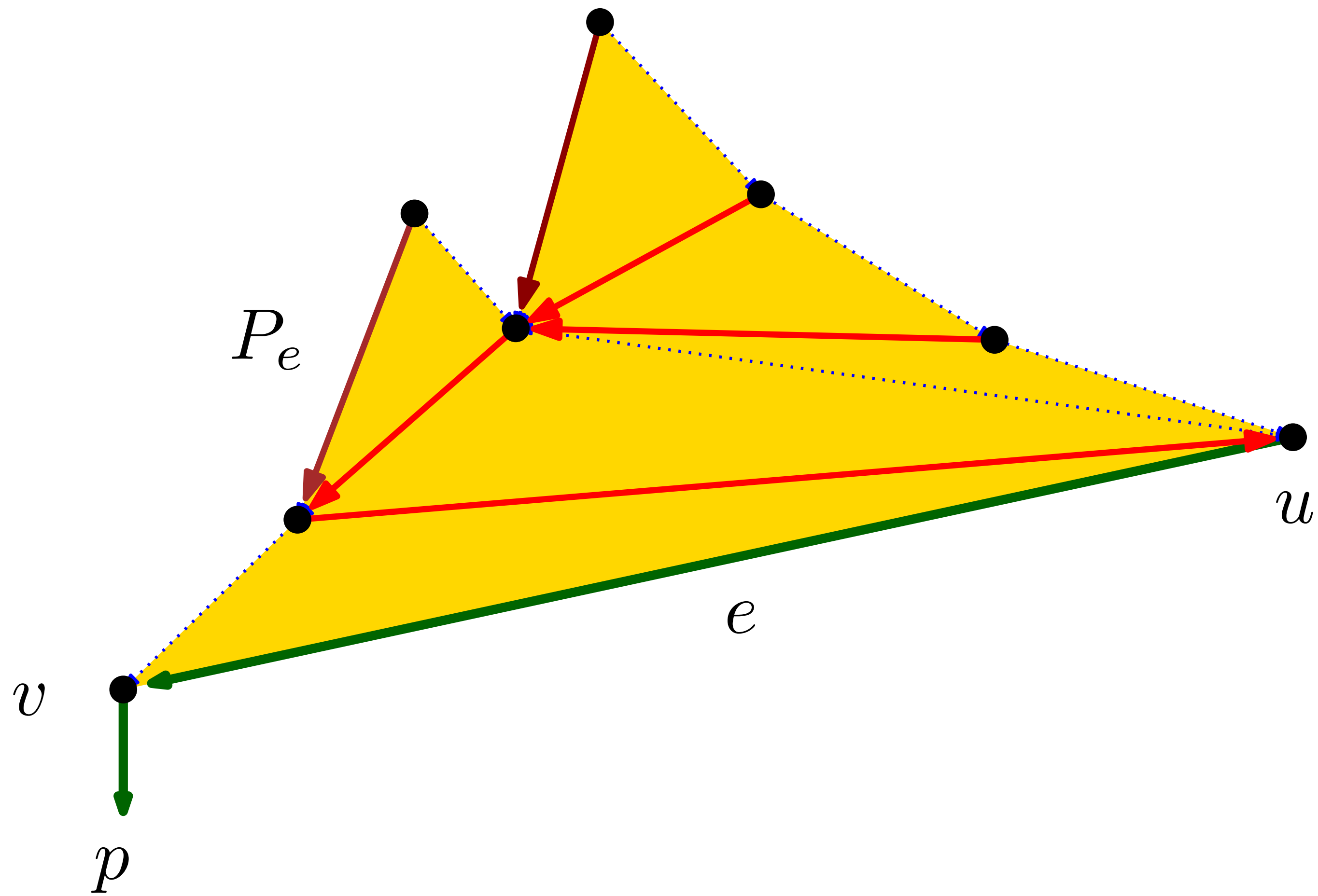

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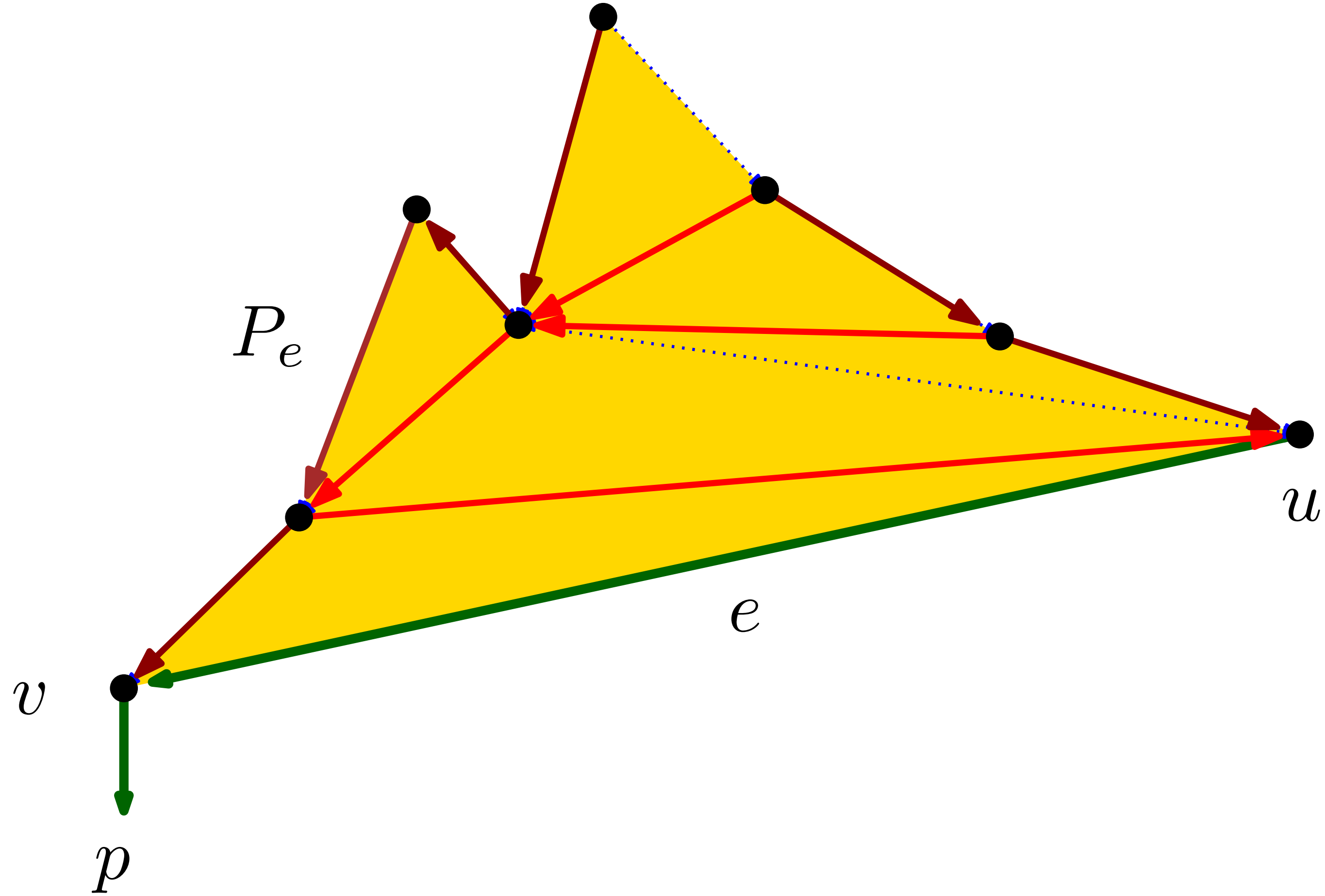

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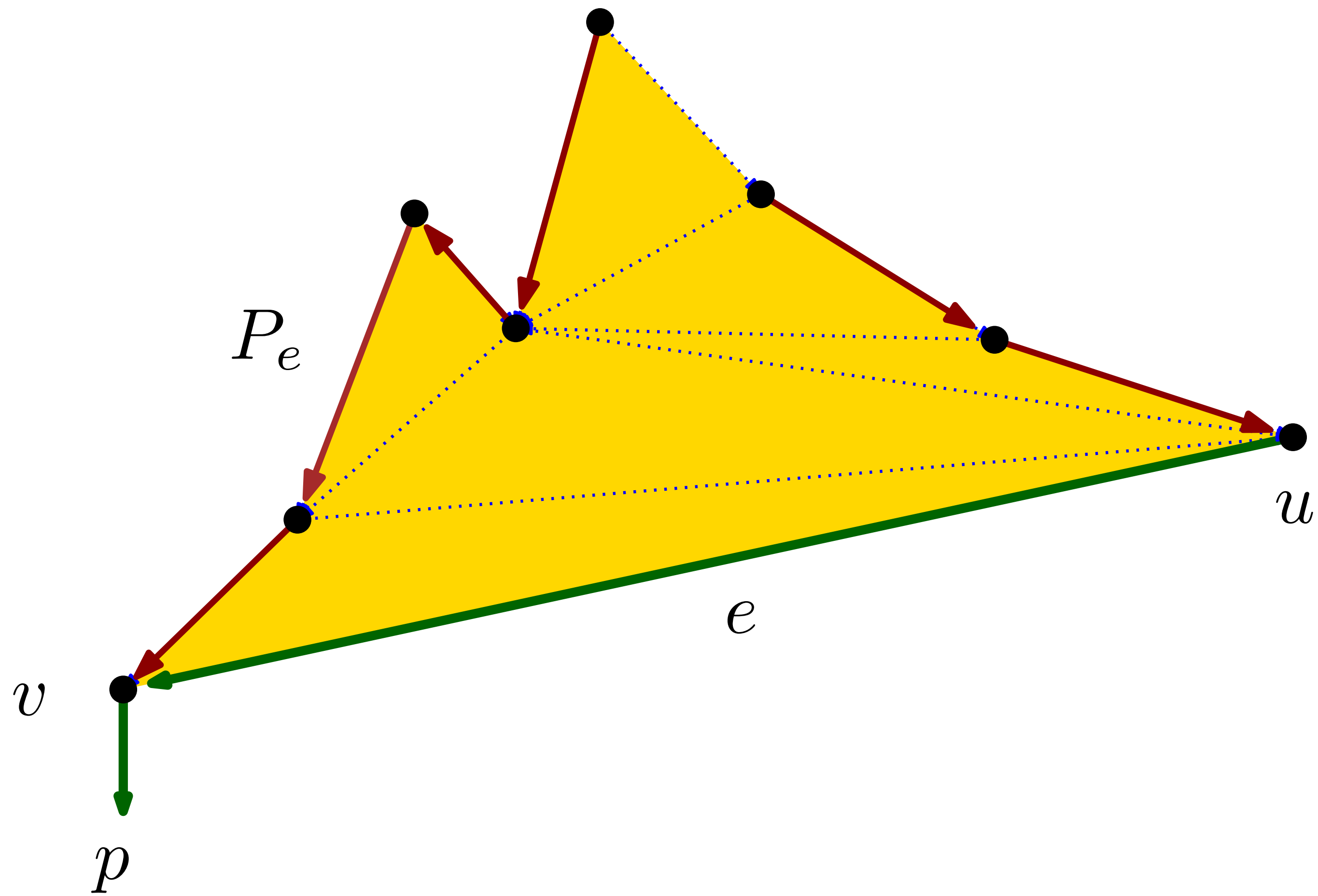

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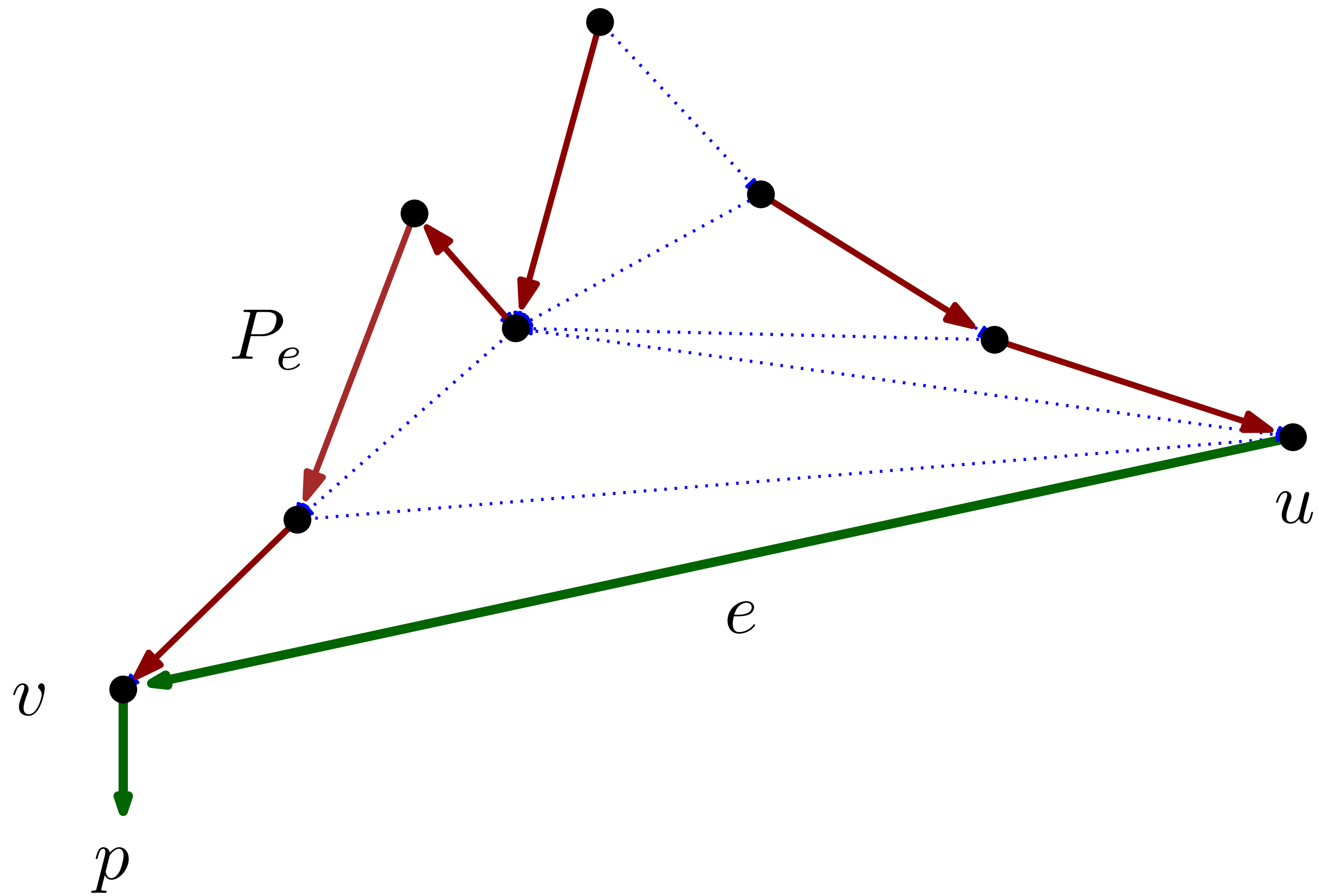

Spanning Trees — Elementary Operations

All five operations define *connected* transition graphs for every point set in general position.

Operation	Single Operation Upper Bound	Single Operation Lower Bound
Exchange	$2n-4$	$\lfloor \frac{3n}{2} \rfloor - 5$ [HHM+99]
Compatible Ex.	$2n-4$	$\lfloor \frac{3n}{2} \rfloor - 5$
Rotation	$2n-4$ [AF96]	$\lfloor \frac{3n}{2} \rfloor - 4$
Empty-Tri. Rot.	$O(n \log n)$	$\lfloor \frac{3n}{2} \rfloor - 4$
Edge Slide	$O(n^2)$ [AR07]	$\Omega(n^2)$ [AR07]

Current upper and lower bounds for the diameter

Spanning Trees — Simultaneous Operations

Upper and lower bounds for the diameter under *simultaneous* operations.

Operation	Simultaneous Upper Bound	Simultaneous Lower Bound
Exchange	1	1
Compatible Ex.	$O(\log n)$ [AAH02]	$\Omega(\frac{\log n}{\log\log n})$ [BRU+09]
Rotation	$O(\log n)$	$\Omega(\frac{\log n}{\log\log n})$
Empty-Tri. Rot.	$8n$	$\Omega(\log n)$
Edge Slide	$O(n^2)$ [AR07]	$\Omega(n)$

Convex Position		
Empty-Tri. Rot.	4	3
Edge Slide	$O(\log n)$	$\Omega(\log n)$

Reconstuct Crossings from Plane Spanning Trees

$S =$ set of n points in general position in $\mathbb{R}^2$,
$\mathcal{T}(S) =$ set of plane spanning trees on S.
$K(S) =$ complete geometric graph on S.

Keller & Perles [2016]: Given the exchange graph on $\mathcal{T}(S)$, for some point set S, one can compute the *intersection graph* of the edges of $K(S)$. In other words, the exchange graph determines which pairs of edges of $K(S)$ cross.

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Oropeza & T. [2018]: Given the *compatible exchange graph* on $\mathcal{T}(S)$, for some point set S, one can compute the *intersection graph* of the edges of $K(S)$.
In other words, the *compatoble exchange graph* determines which pairs of edges of $K(S)$ cross.

Open Problems

Improve the diameter bounds for the "tree graphs."

- Are $\lfloor \frac{3}{2} n \rfloor$ **exchange** operations enough to transform a plane spanning tree to any other plane spanning tree?
- Is the diameter for **empty-triangle rotation** $O(n)$?
- Is the diameter for **simultanous edge slides** $\Theta(n)$, or $\Theta(n^2)$, or something in between?

Transformation graphs for other variants:

- Is the space of plane spanning trees of **max degee**$\leq k$ connected under any or all of the five operations?
- If the edges have unique labels, can these operations "shuffle" the labels arbitrarily?

Open Problems

Reconstruction of intersection pattens from “tree graphs.”

- Does the transition graph of **rotations** contain enough informartion to reconstruct the intersection graph of the edges of $K(S)$?
- For finding a possible *counterexample*, we need to generate finite point sets $S_1, S_2 \subset \mathbb{R}^2$ such that $|S_1| = |S_2|$, $|\mathcal{T}(S_1)| = |\mathcal{T}(S_2)|$, and the intersection patterns of $K(S_1)$ and $K(S_2)$ are different.

Thank you for your attention!