Conflict-Free coloring of string graphs

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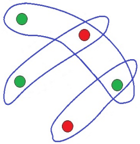
classic Non-Monochromatic (proper) coloring for Hypergraphs

Definition (proper coloring)

Given a hypergraph $H = (V, \mathcal{E})$, a-coloring $c : V \to \{1, \ldots, r\}$. is called proper if $\forall e \in \mathcal{E}, |e| \geq 2,\ \exists v, v' \in e$ s.t. $c(v) \neq c(v')$. $\chi(H)$ denotes the chromatic number of H .

Definition (CF-coloring)

Given a hypergraph $H = (V, \mathcal{E})$, a-coloring $c : V \to \{1, \ldots, r\}$ *CF-coloring* *if* $\forall e \in \mathcal{E}\ \exists v \in e$ *s.t.* $c(v) \neq c(v')\ \forall v' \in e \setminus \{v\}$.
$\chi_{CF}(H)$ *denotes the CF-chromatic number of* H .



A CF-coloring of a hypergraph with 2 colors.

Conflict-Free coloring for Hypergraphs

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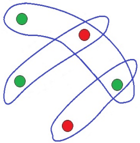

A CF-coloring of a hypergraph with 2 colors.

Definition (k-CF cloring)

If in each hyperedge some color appears $1 \leq i \leq k$ then c is a k-CF coloring. Denote by $\chi_{k-CF}(H)$ the ..

Theorem (Pach and Tardos, '09)

$\forall H$ with m hyperedges $\chi_{CF}(H) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}} = O(m^{\frac{1}{2}})$ and this is tight.

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Theorem (S '07)

Let $H = (V, E)$ be a hypergraph on n vert. $t > 1$ a fixed integer. If $\chi(H') \leq t \; \forall H' \subset H$.
Then

$$\chi_{CF}(H) = O(t \log n).$$

Asymptotically tight for constant t.

k-CF-coloring for hypergraphs

Theorem (Keller, Rok, S. '18+)

Any hypergraph with n vertices and m hyperedges $\chi_{k-CF}(H) = O(m^{\frac{1}{k+1}} log^{\frac{k}{k+1}} n)$ and its near optimal.

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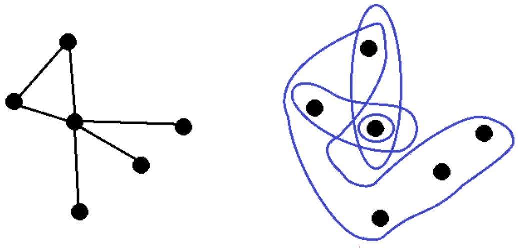
Observation (for the lower bound)

The complete $k+1$-uniform hypergraph on n vert has $m = \binom{n}{k+1}$ hyperedges. Needs at least $\frac{n}{k} = \Omega(m^{\frac{1}{k+1}})$ colors in any $k - CF$-coloring.

Neighborhood hypergraph of a graph

Definition

Given a graph $G = (V, E)$ the open (resp. closed) neighborhood hypergraph of G is the hypergraph $H = (V, \mathcal{E})$, where $\mathcal{E} = \{N(v) : v \in V\}$ (resp. $\mathcal{E} = \{N(v) \cup \{v\} : v \in V\}$).



Drawing of a graph and its open neighborhood hypergraph.

Open/closed CF-coloring of *graphs*

Definition

An open (resp. close) k-CF-coloring of G is a k-CF-coloring of its open (resp. closed) neighborhood hypergraph.
Let $\chi^{on}_{k-CF}(G)$ (resp. $\chi^{cn}_{k-CF}(G)$) denote the corresponding $k-CF$ chromatic numbers.
When $k=1$ simply write $\chi^{on}_{CF}(G)$ (resp. $\chi^{cn}_{CF}(G)$).

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Later extended:

Theorem (Pach and Tardos, '09)

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closed CF-coloring of *graphs*

Observation

$\forall G$ *we have* $\chi_{CF}^{cn}(G) \leq \chi(G)$.

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$\forall$ *graph* G *on* n *vertices* $\chi^{cn}_{CF}(G) \leq 2\chi^{on}_{CF}(G)$
Moreover,

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Theorem (Glebov, Szabó, Tardos '14)

$\exists G$ *on* n *vertices with* $\chi_{CF}^{cn}(G) = \Omega(\log^2 n)$.

Special graphs

Theorem (closed coloring Abel et al. '17)

(i) For a planar G $\chi^{cn}_{CF}(G) \leq 3$*. Bound is tight!*
(ii) If G *does not contain a* K_{r+1} *as a minor then* $\chi^{cn}_{CF}(G) \leq r$*.*

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Let G be an intersection graph of unit disks. Then $\chi^{cn}_{CF}(G) \leq 6$

Theorem (open coloring Keller, S '17)

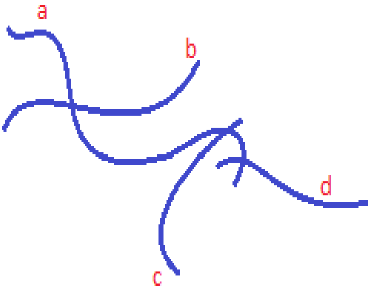
Let G be an intersection graph of n pseudo-disks. Then $\chi^{on}_{CF}(G) = O(\log n)$. Asymptotically tight.

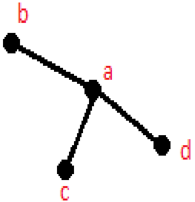
A beautiful follow-up result by Keszegh 13-june-2018 10:00. Be there or be ...... (a psudo-disk ?)

String graphs

Definition

A string graph is a graph s.t. vertices are curves and ∃ an edge between 2 curves iff they intersect.





A family of 4 curves and its intersection graph.

String graphs

Theorem (Keller, Rok, S. , '18+)

Any hypergraph with n vertices and m hyperedges can be k-CF-colored with $O(m^{\frac{1}{k+1}} \log^{\frac{k}{k+1}} n)$ colors.

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Corollary (open coloring Keller, Rok, S. '18+)

If $F^k(n)$ denotes the maximum open k-CF chrom. numb. of a string graph on n v. then $\Omega(n^{\frac{1}{k+1}}) = F^k(n) = O(n^{\frac{1}{k+1}} \log^{\frac{k}{k+1}} n)$.

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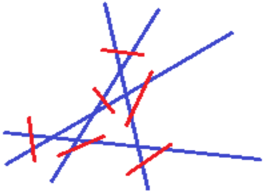

Example with CF-coloring, i.e., $k = 1$: There are i blue segments and $\binom{i}{2}$ red segments. $n = i + \binom{i}{2}$. At least i colors are needed for open CF-coloring.

So .... what interesting can be said?

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Theorem (open coloring Keller, Rok, S. '18+)

$\forall$ string graph G on n vert. if $\chi(G) \leq t$ then
$\chi_{CF}^{on}(G) = O(t^2 \log n)$.
Asymptotcially tight for any constant $t \geq 2$.

There exists families of n L-shapes with open CF-chromatic number $\Omega(\sqrt{n})$.

Special string graphs: open coloring L-shapes

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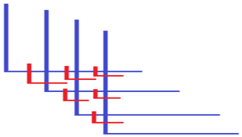

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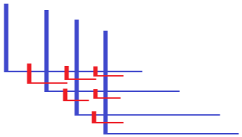

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Theorem (Keller, Rok, S. '18+)

Families of n L-shapes have open 2-CF-chromatic number $O(\log n)$.

In fact, one can obtain k-CF-colorings for small k for many different families of n "simple" shapes with $O(\log n)$ colors.

Special string graphs: open coloring intervals overlap graphs

Theorem (Keller, Rok, S. '18+)

If $I(n)$ denotes the maximum open CF-chromatic number of an interval overlap graphs on n vertices, then $\Omega(\log n) = I(n) = O(\log^2 n)$.

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Theorem (Keller, Rok, S. '18+)

If $L(n)$ denotes the maximum open CF-chromatic number of a family of n grounded L-shapes, then $\Omega(\log n) = L(n) = O(\log^3 n)$.

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Conjecture

Given a fixed curved c any family $\mathcal{F}$ of n curves each intersecting c ≥ 1 and $\leq$ a cnst numb. of times can be open CF-colored with polylog(n) colors.


Thank You!