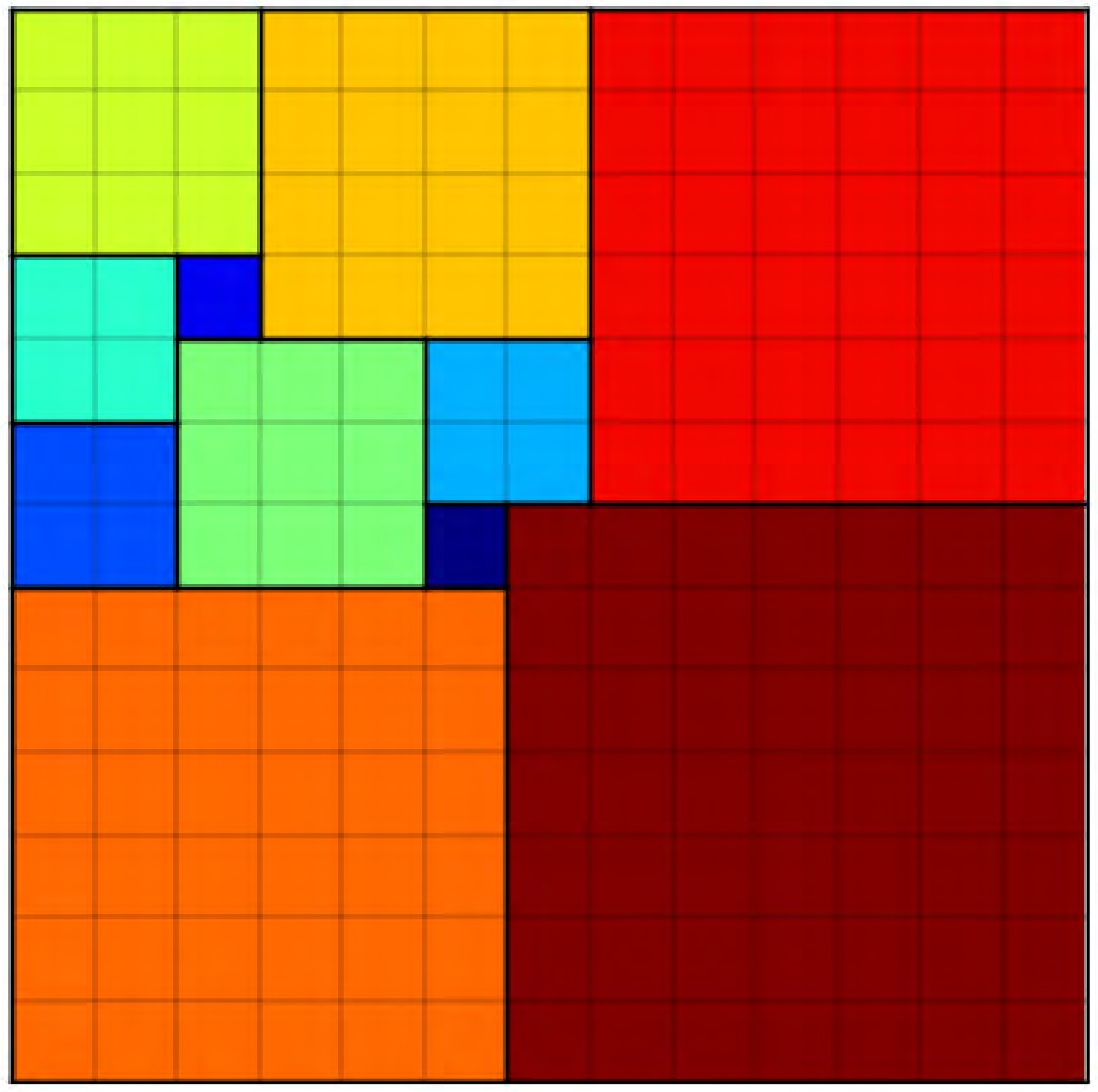
TRIANGLES: ERDŐS, TUTTE, AND BUTTERFLIES

Andrey Kupavskii,
MIPT, Moscow

SQUARING THE SQUARE

Problem (Erdős 1934)

Is it possible to tile a square with finitely many smaller squares, no two of which are congruent?



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Theorem (Brook, Smith, Stone, Tutte 1940; Sprague)

Yes! Smallest such tiling consists of 21 squares.

   

THE DISSECTION OF RECTANGLES INTO SQUARES

By R. L. Brooks, C. A. B. Smith, A. H. Stone and W. T. Tutte

J. H. Hunter-Tod, W. F. Campbell, T. Oates, F. C. Strachan, H. C. Corben, A. J. Skinner, W. E. Blundell, S. H. Moss.
H. C. Schwab, N. J. P. Hutchison, J. M. Tasker, G. C. C. Chivers, H. Bondi, J. C. Pijper, D. S. Palmer, B.A., G. P. S. Streatfeild, C. A. B. Smith, J. H. Wilkinson, W. T. Tutte.
D. C. Smith, E. Wild, A. R. Stokes, S. N. Higgins, D. T. Copley, B.A., S. Rosenbaum, A. Nisbet, A. K. Weaver, F. J. Patterson, A. H. Stone, C. W. Parkinson, O. Kempthorne.
C. H. Elsey-Warren *(Secretary)*, W. J. Corlett *(Vice-President)*, R. L. Brooks, A. S. Besicovitch, F.R.S., F. J. Anscombe *(President)*, W. R. Dean, M.A., C. A. Coulson, Ph.D., H. M. Cundy, B.A., D. S. Evans, B.A.

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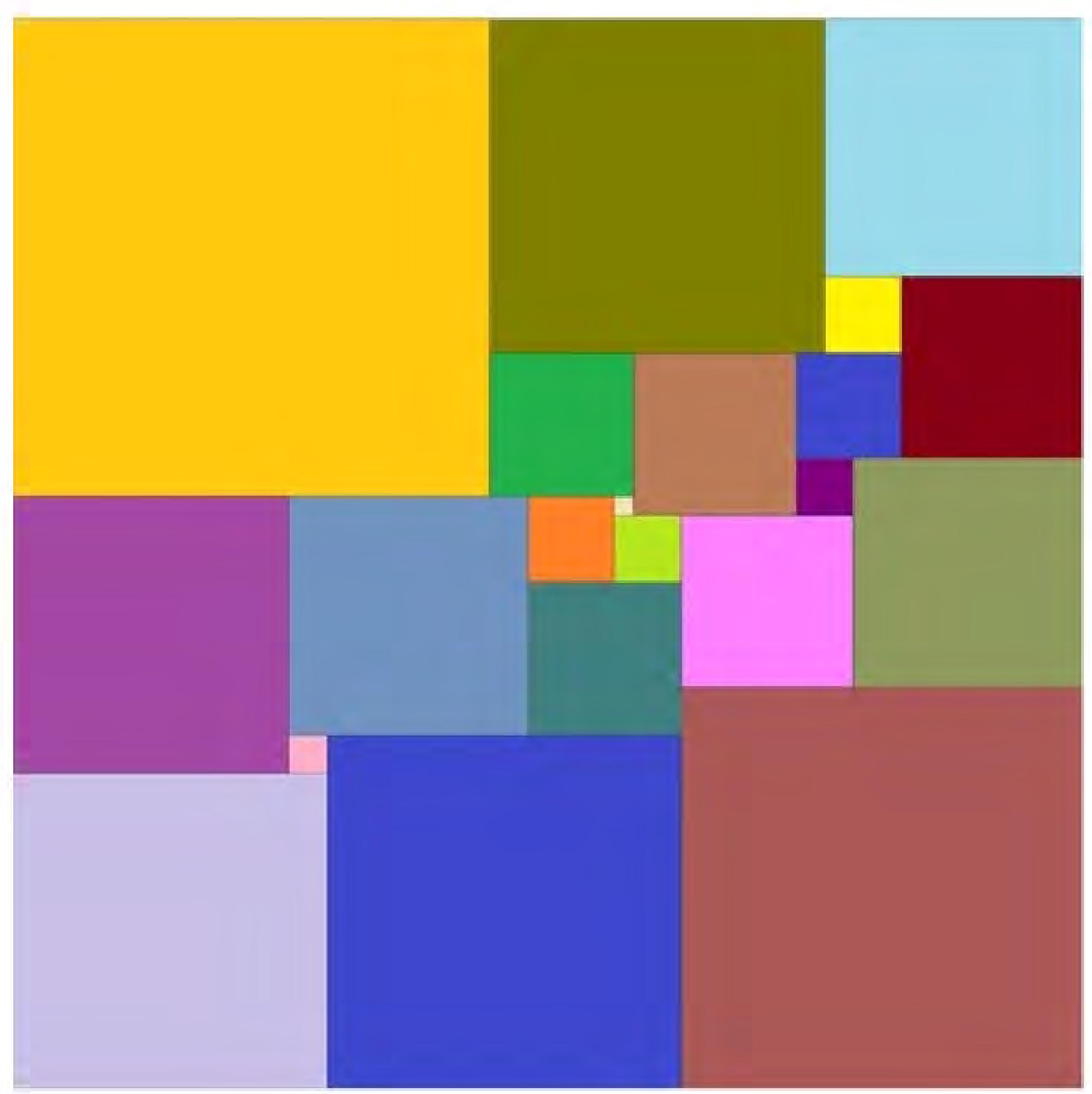
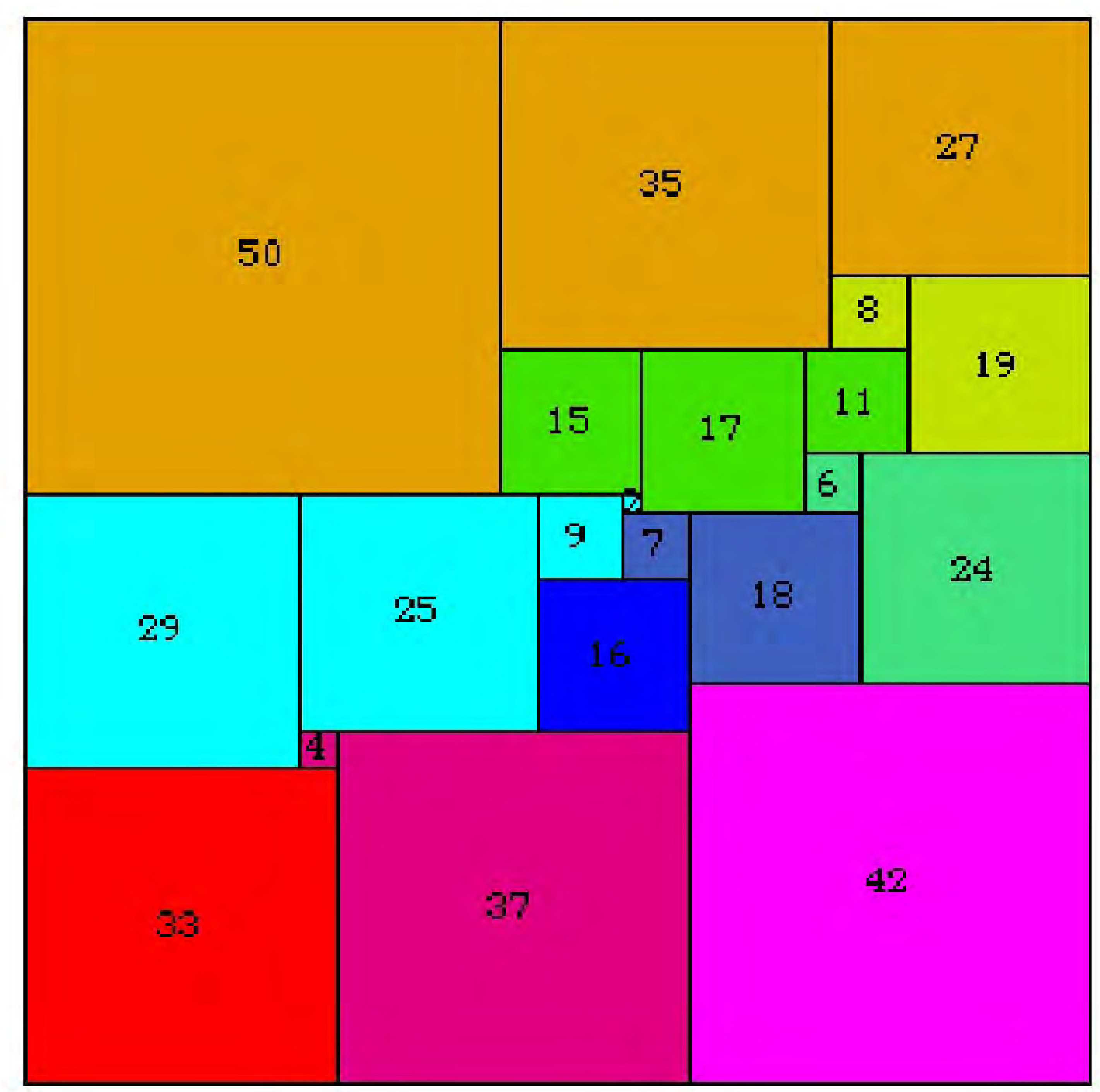
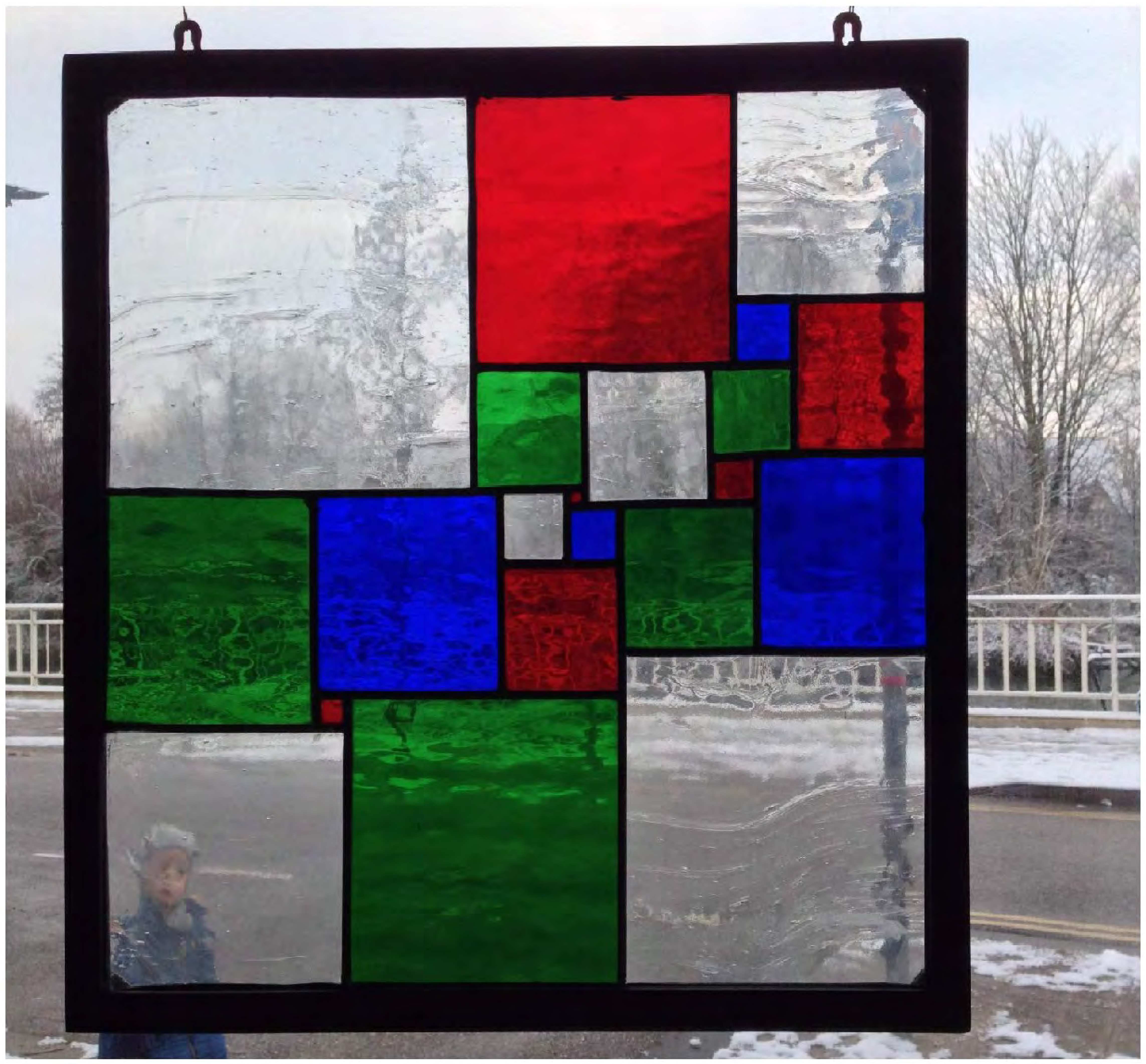
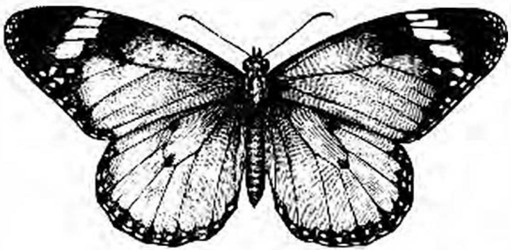




SQUARING THE SQUARE



The Butterfly Effect



Edward Lorenz (1917-2008)

"like many of Erdős's other casual conjectures, it would change the lives of those who worked on it. One of them, Cedric Smith, would later remark with some—only slightly strained—justification that, much as the flapping of a butterfly's wing in Montana might have caused a monsoon in India, Erdős's little conjecture might have altered the fate of Western civilization." **(Bruce Schechter, 1998)**

SQUARING THE SQUARE

Problem (Erdős 1934)
Is it possible to tile a square with finitely many smaller squares, no two of which are congruent?

Theorem (Brook, Smith, Stone, Tutte 1940; Sprague)
Yes! Smallest such tiling consists of 21 squares.

Corollary. The plane can be tiled by infinitely many pairwise noncongruent squares whose side lengths are bounded from below by a constant $c > 0$.

TILING WITH EQUILATERAL TRIANGLES

Theorem (Tutte 1948)

An equilateral triangle cannot be tiled with finitely many pairwise noncongruent equilateral triangles.

Problem (Nandakumar 2016)

Is it possible to tile the plane with pairwise noncongruent equilateral triangles (whose side lengths are bounded from below by a constant $c > 0$)?

Theorem (Richter 2012)

There exists a tiling of the plane with pairwise noncongruent equilateral triangles.

Theorem (P., Tardos 2018; Richter, Wirth) There is no tiling of the plane with pairwise non-congruent equilateral triangles whose side lengths are bounded from below by a constant $c > 0$.

$$\Uparrow$$

Theorem. Let $\mathcal{T}$ be a tiling of the plane with equilateral triangles of side lengths $\geq c > 0$ such that no two triangles share a side.

Then the triangles in $\mathcal{T}$ have at most three different side lengths, a, b, c with $a = b + c$, and the tiling is periodic.

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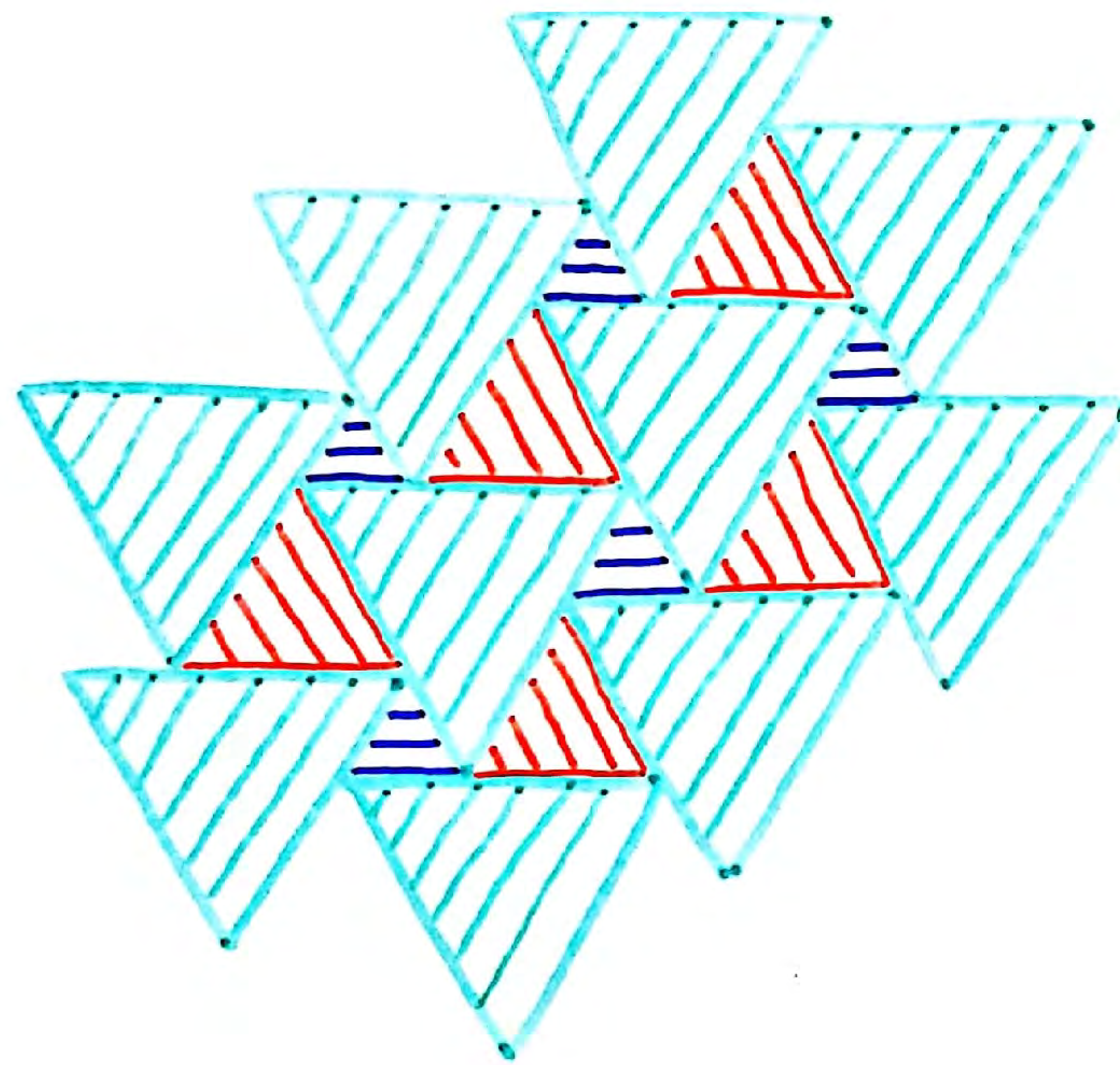
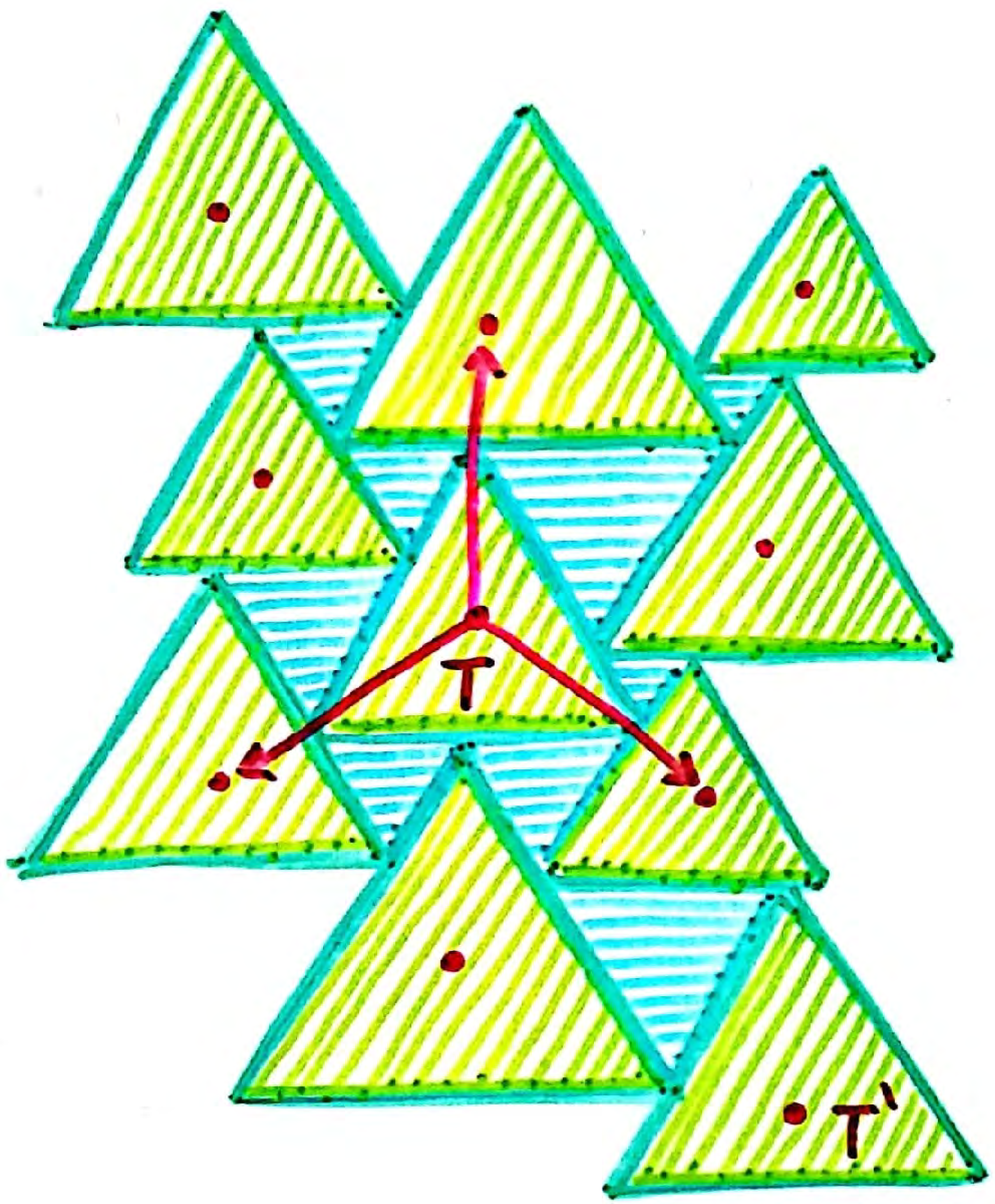


HARMONIC FUNCTIONS / RECURRENT WALKS



$s(T) < s(T')$

Directed graph on large triangles, out-degree = 3

side length of $\forall$ large = $\frac{1}{3} \sum$ side lengths of out-neighbors

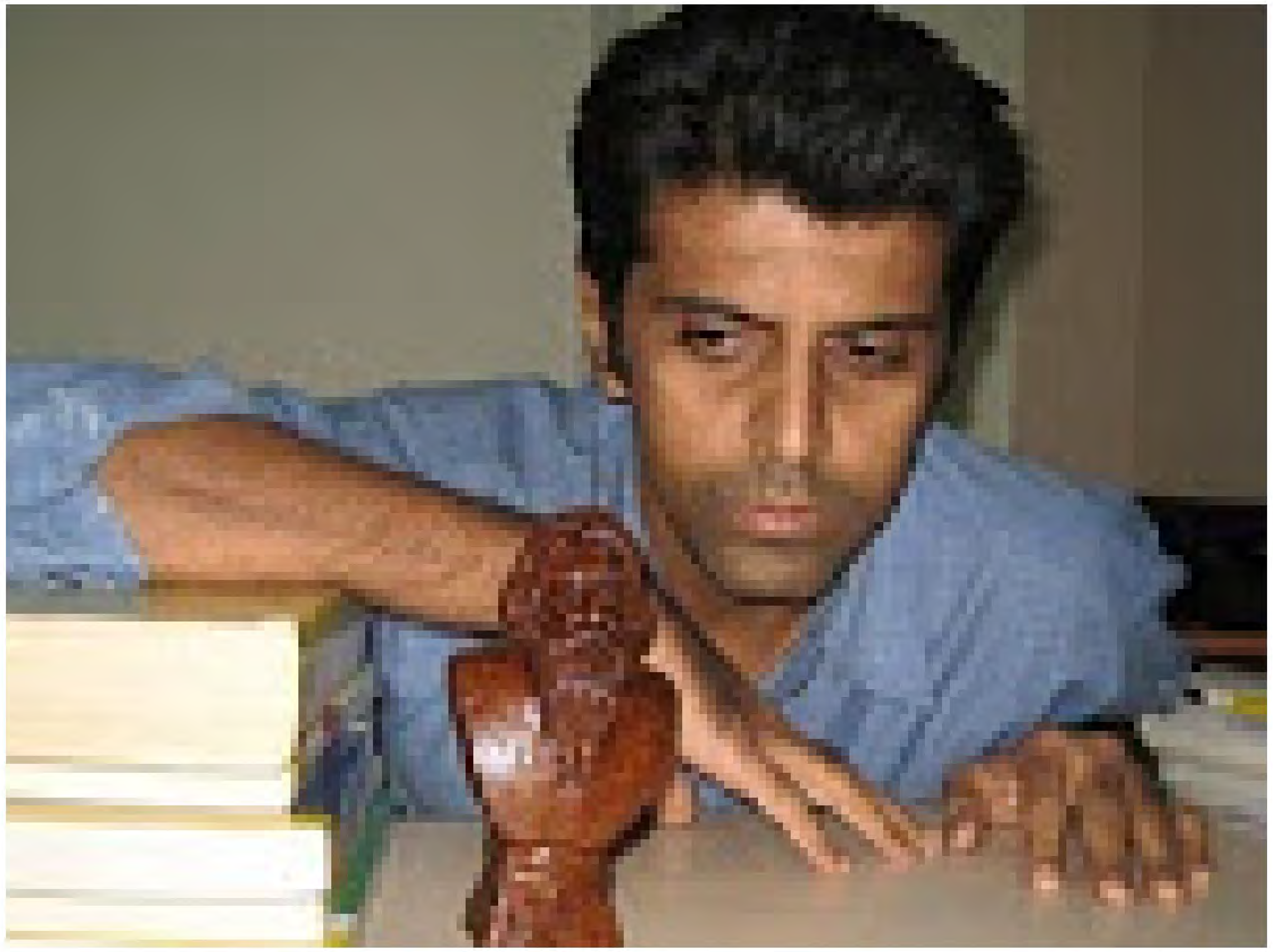
Random walk on large triangles: go to an out-neighbor with probability $\frac{1}{3}$

Recurrent walk

$T = T_0, T_1, T_2, \ldots, T_N = T'$

$s(T_i)$, $i = 0, 1, 2, \ldots$ martingale

R. NANDAKUMAR'S PROBLEMS



computer programmer,
college teacher,
Kochi (India)

R. NANDAKUMAR'S PROBLEMS

- **Conjecture.** For every $n \geq 2$, any convex body in the plane can be partitioned into n convex pieces of equal area and equal perimeter.

$n = p^k$

Karasev, Hubard, Aronov 2014

Blagojević, Ziegler 2014

Any n:

Akopyan, Avvakumov, Karasev 2018

R. NANDAKUMAR'S PROBLEMS

- **Conjecture.** For every $n \geq 2$, any convex body in the plane can be partitioned into n convex pieces of equal area and equal perimeter.

- **Problem.** Is it possible to tile the plane with pairwise noncongruent triangles of equal area and equal perimeter?

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A Negative Answer

Problem. Is it possible to tile the plane with pairwise noncongruent triangles of equal area and equal perimeter?

In such a tiling, no two triangles share a side.

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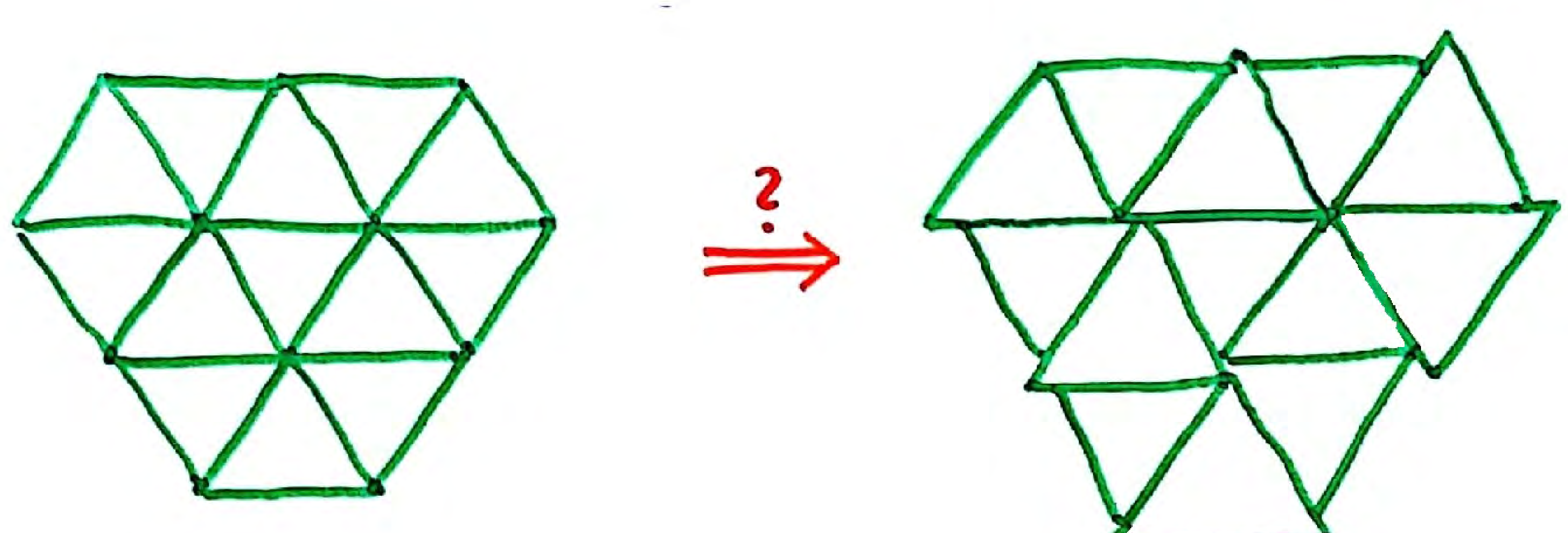
First attempt: Can one perturb the regular tiling to avoid that two triangles share a side ?



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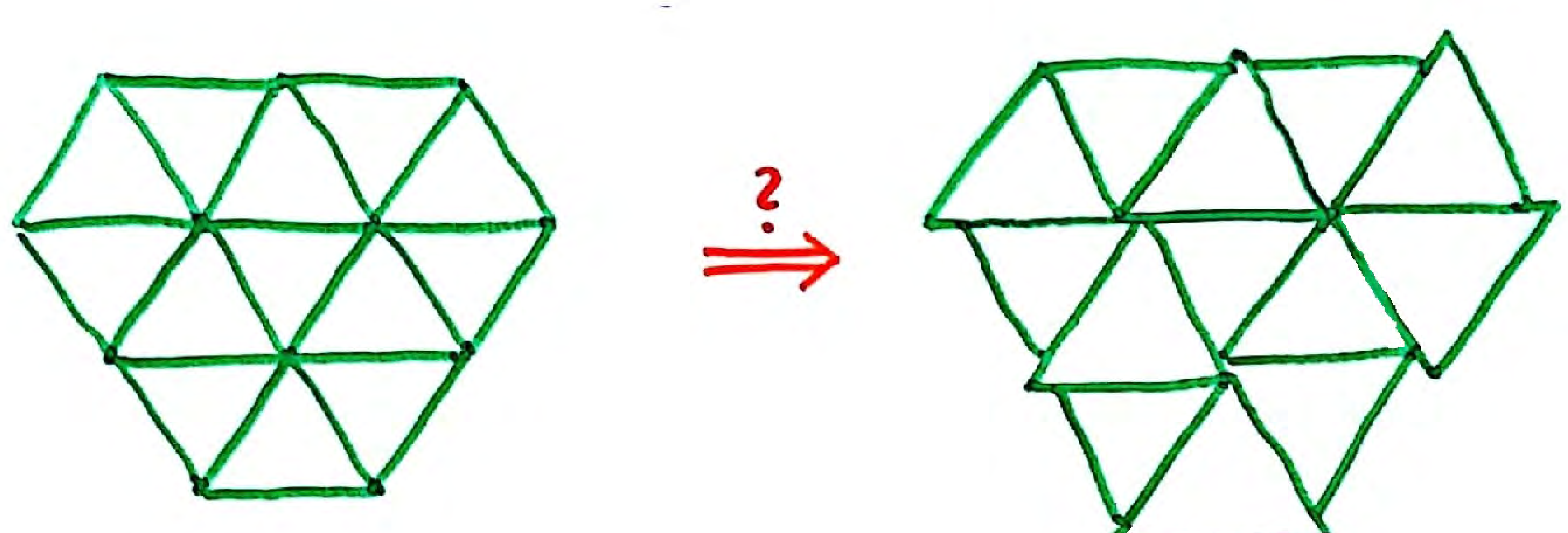

NO!

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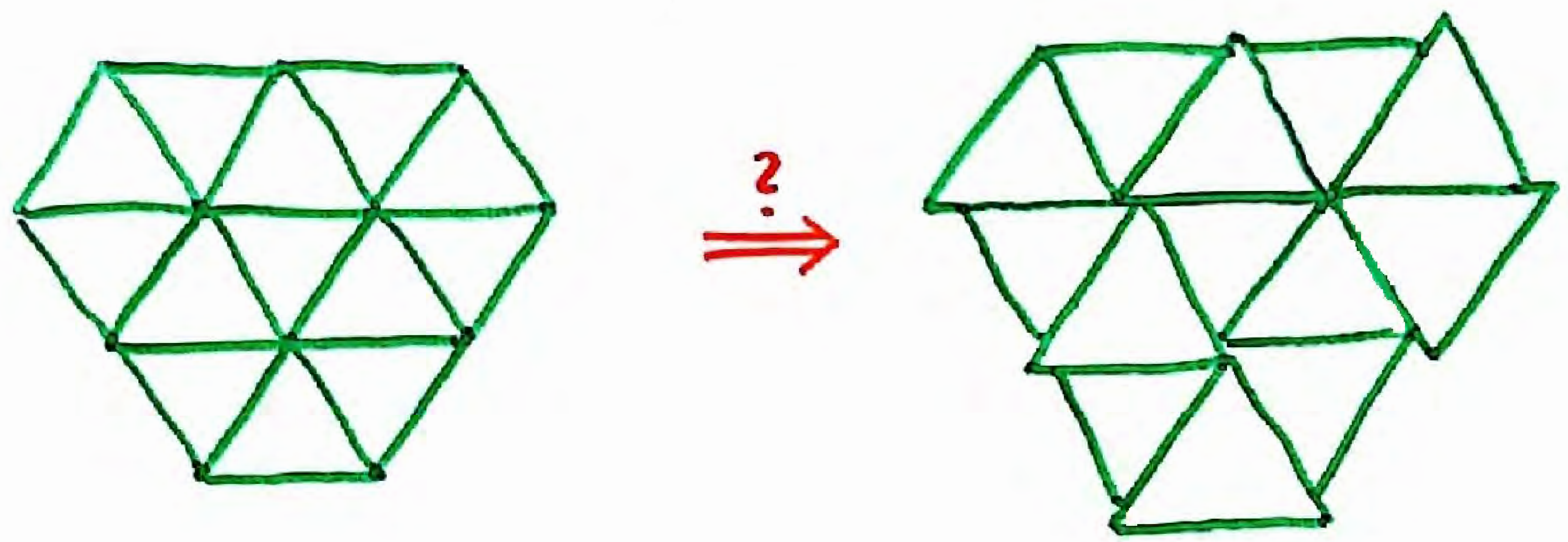

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Theorem. Let $\mathcal{T}$ be a locally finite tiling of the plane with triangles, all sides of which belong to $[1,2)$. Then there are two triangles in $\mathcal{T}$ that share a side.

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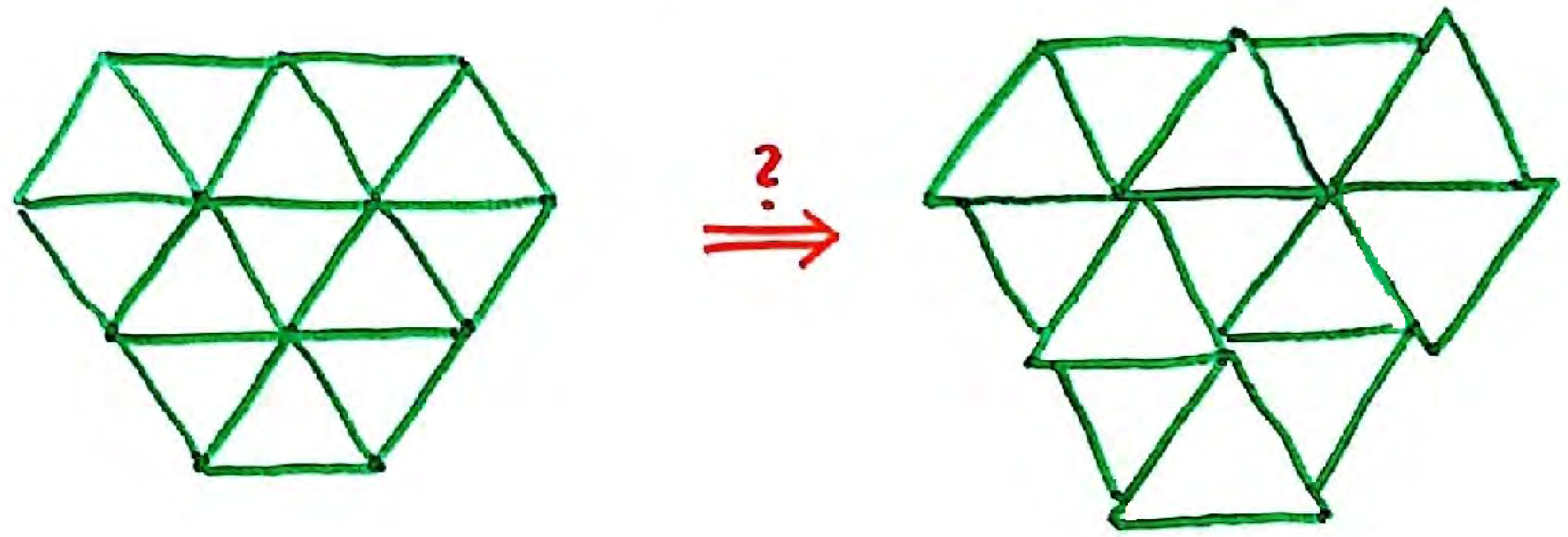


NO!

Theorem. Let $\mathcal{T}$ be a locally finite tiling of the plane with triangles, all sides of which belong to $[1,2)$. Then there are two triangles in $\mathcal{T}$ that share a side.

Theorem. In any locally finite tiling of the plane with triangles, there is a triangle that has a side which is the union of one or more sides of other triangles.

A Negative Answer

Problem. Is it possible to tile the plane with pairwise noncongruent triangles of equal area and equal perimeter?

Theorem (Kupavskii, P., Tardos 2018)
There is no tiling of the plane with pairwise noncongruent triangles of equal area and equal perimeter.

A Negative Answer

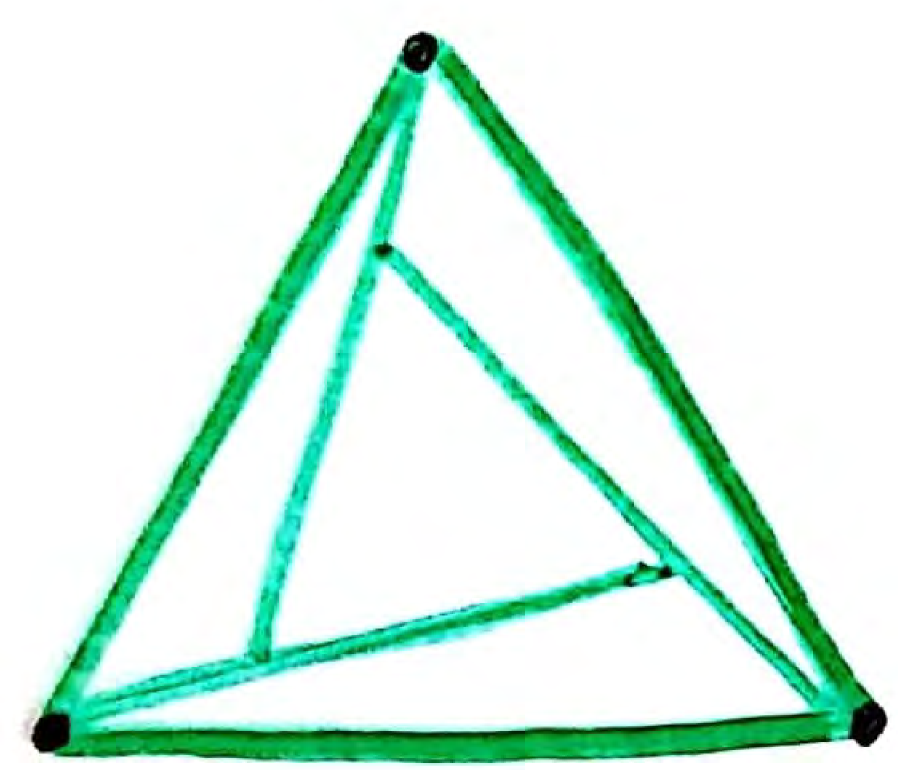
Theorem (Kupavskii, P., Tardos 2018)
There is no tiling of the plane with pairwise noncongruent triangles of equal area and equal perimeter.

In such a tiling, no two triangles share a side.

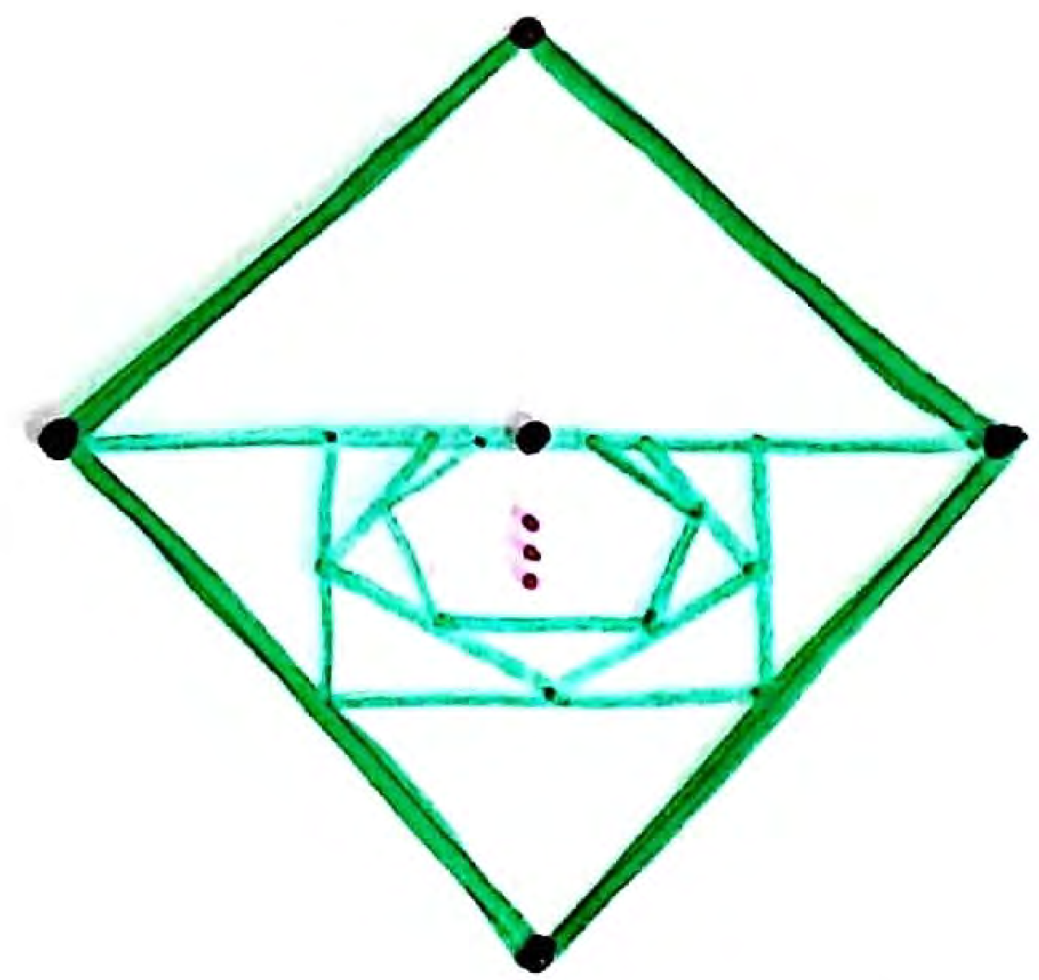
Theorem. Let $\mathcal{T}$ be a tiling of the plane with triangles of equal perimeter, each of which has area $\geq \varepsilon > 0$.
Then there are two triangles in $\mathcal{T}$ that share a side.

Theorem. (Kupavskii, P., Tardos 2018)

For $k \geq 4$, there exists no tiling of a convex k-gon with finitely many triangles, no two of which share a side.

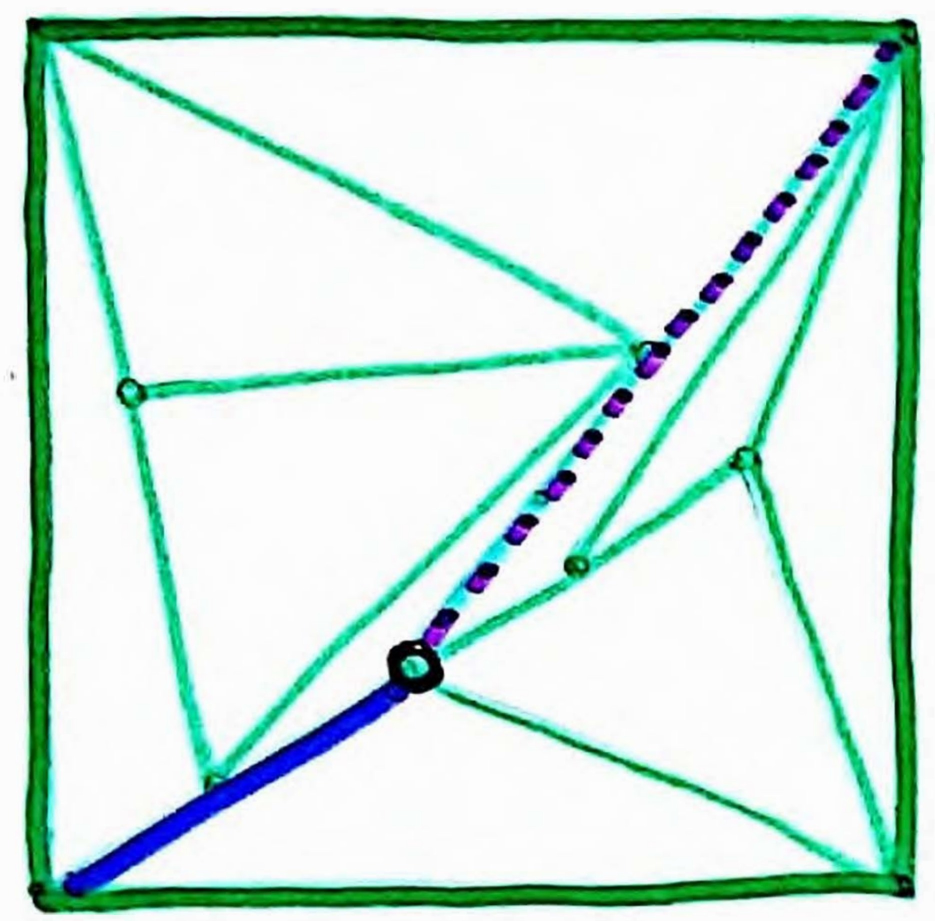


$k=3$



infinite

Proof : stretch = minimal segment decomposable into sides in 2 different ways



By Euler's Polyhedral Formula,

$$\underset{\text{subdividing}}{v_{bd} + 2v_{int} - v^*_{int}} = \underset{\text{faces (triangles)}}{t + 2}$$

Since every stretch contains ≥ 3 sides,

$$v^*_{int} \geq \frac{3t - v_{bd}}{3}$$

Combining,

$$v_{bd} + 3(v_{int} - v^*_{int}) \leq 3 \Rightarrow \begin{array}{l} k = 3 \\ v_{int} = v^*_{int} \end{array}$$

A Positive Answer

Theorem (Kupavskii, P., Tardos 2018; Frettlöh 2018)

There exist tilings of the plane with noncongruent

1. unit perimeter triangles, each of which has area $\geq \varepsilon > 0$;

2. unit area triangles, each of which has perimeter $\leq C$.