$\mathbb{Z}_2$-embeddings and Tournaments

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Institute of Science and Technology

June 12, 2018

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Formally, $D(e)$ is injective for every edge,
$C = \{\mathbf{p} \in S : \ |D^{-1}[\mathbf{p}]| > 1\}$ is finite, and every $\mathbf{p} \in C$ is a proper edge crossing of exactly **two edges**.

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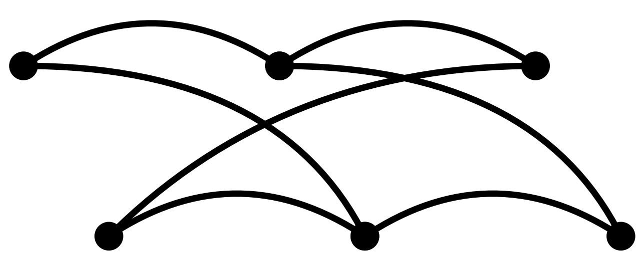

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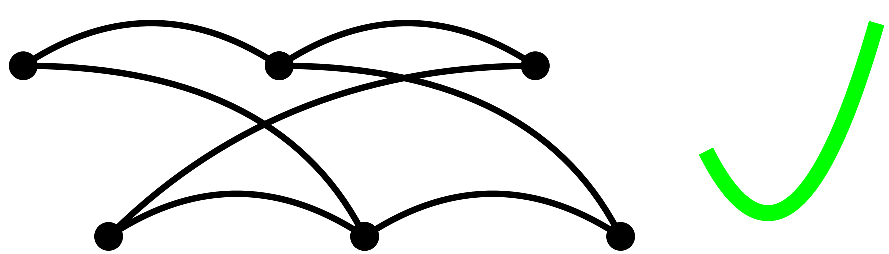

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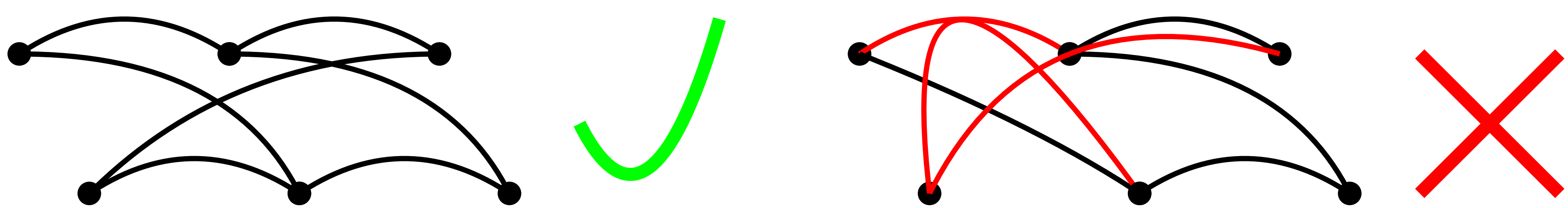

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Injective D is an **embedding**.

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Let $I_{D(G)} = \{\{e, f\} \in \binom{E}{2} | \; e \cap f = \emptyset \; \& \; |D(e) \cap D(f)| =_2 1\}$.

A drawing for which $I_{D(G)} = \emptyset$ is a $\mathbb{Z}_2$-**embedding**.

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Theorem 2 (Cairns and Nikolayevsky 2000, Pelsmajer, Schaefer, and Štefankovič 2009)**.** *If a graph G admits a strong $\mathbb{Z}_2$-embedding on S then G can be embedded on S.*

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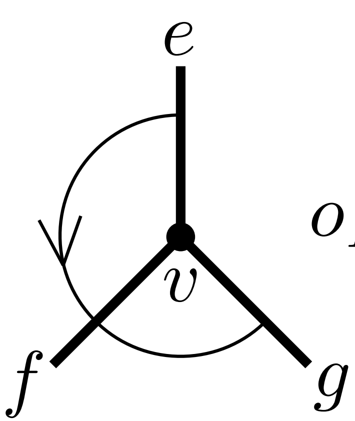

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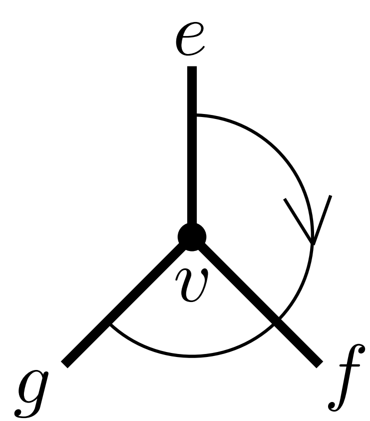

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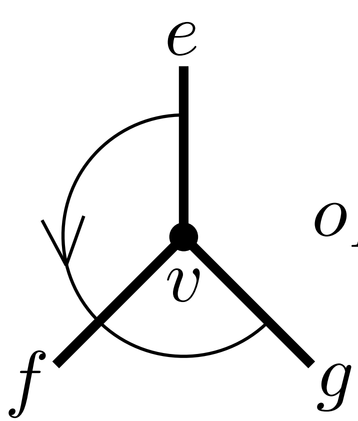

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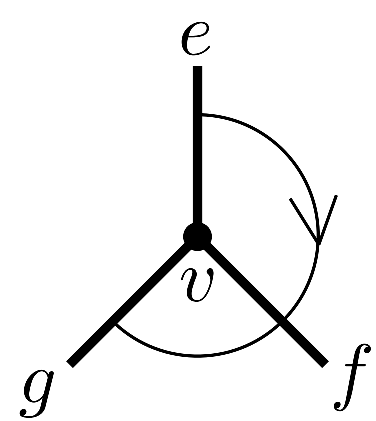

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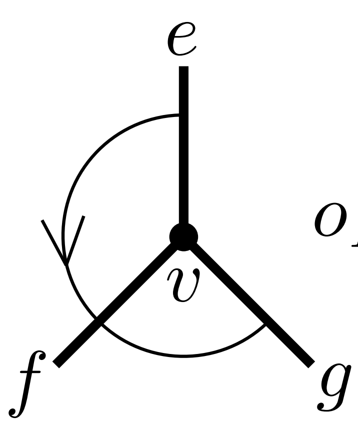

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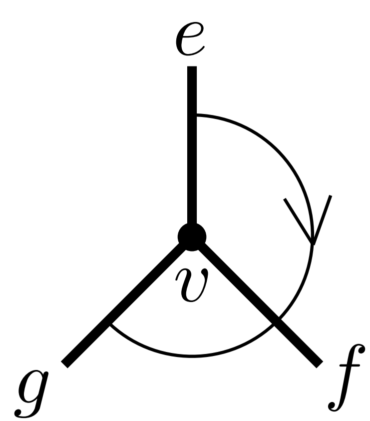

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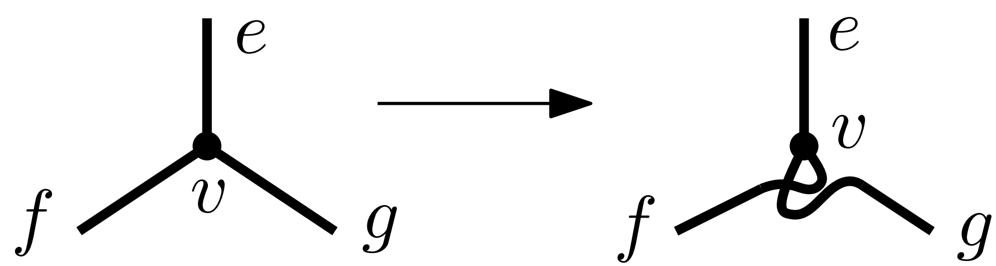

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We count the number of 3 element subsets of $\{e, f, g, h \ni v\}$ for which σ_D and o_D return the same value.

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Proof. We count the number of 3 element subsets for which $\mathrm{cr}(\{e_1, e_2, e_3\}) =_2 0$. Thus, we count the number of triples of vertices in a graph with 4 vertices inducing an even number of edges. This number must be even. □

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Proof. Let $G_{aux}(v) = (\delta(v), E')$, where $ef \in E'$, if $|D(e) \cap D(f)| =_2 1$. $G_{aux}(v)$ must be a complete bipartite graph. Pushing every edge in one part over v renders $G_{aux}(v)$ empty. □

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Proof. Obviously, $\{\mathbf{e},f,g\} \in \clubsuit_D(v)$ iff it appears as an element of exactly one summand of Δ. Let $\{e',f,g\}$, $e' \neq \mathbf{e}$. Then by **Claim 1.** applied to $\{\mathbf{e},e',f,g\}$, $\{e',f,g\} \in \clubsuit_D(v)$ iff it appears once or three times. $\square$

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Let $v \in e \in E$. The $\mathbb{Z}_2$-**rotation tournament** is the tournament $T_D(v, e)$ on $\{f \in E : v \in f\} \setminus \{e\}$ s.t. $\overrightarrow{fg}$ if $\sigma_D(e, f, g) = +1$.

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Proof. If $T_D(v, e)$ is acyclic then we obtain D' so that $\sigma_{D'}(e, g, h) = o_{D'}(e, g, h)$ for all $g, h \ni v$. It follows by Claim 1. that $\sigma_{D'}(e', g, h) = o_{D'}(e', g, h)$ for all $e', g, h \ni v$.

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Proof. If $T_D(v,e)$ is acyclic then we obtain D' so that $\sigma_{D'}(e,g,h) = o_{D'}(e,g,h)$ for all $g,h \ni v$. It follows by Claim 1. that $\sigma_{D'}(e',g,h) = o_{D'}(e',g,h)$ for all $e',g,h \ni v$. By the corollary, applied with $\mathbf{e} := f$ and $D := D'$ we obtain $\sigma_D(f,g,h) = \sigma_{D'}(f,g,h) = o_{D'}(f,g,h)$. $\square$

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A drawing D of a graph G is $\mathbb{Z}_2$-**acyclic** if $T_D(v,e)$ is acyclic for all $v \in e \in E$.

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Corollary 2. *If G admits a $\mathbb{Z}_2$-acyclic $\mathbb{Z}_2$-embedding D on a surface S then G admits a strong $\mathbb{Z}_2$-embedding on S.*

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Corollary 4. *If the restrictions of a drawing D of G to all 4-stars of G are $\mathbb{Z}_2$-acyclic then D is $\mathbb{Z}_2$-acyclic.*

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Claim 4. *Every planar $\mathbb{Z}_2$-embedding of a 3-connected graph G is $\mathbb{Z}_2$-acyclic.*

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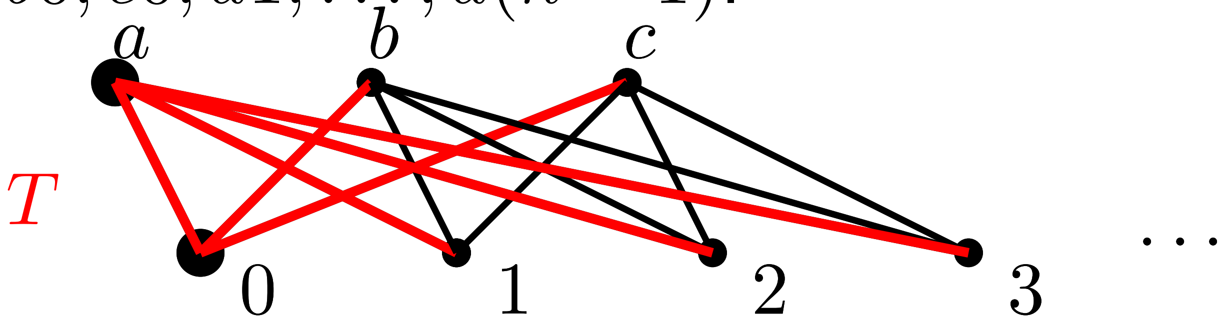

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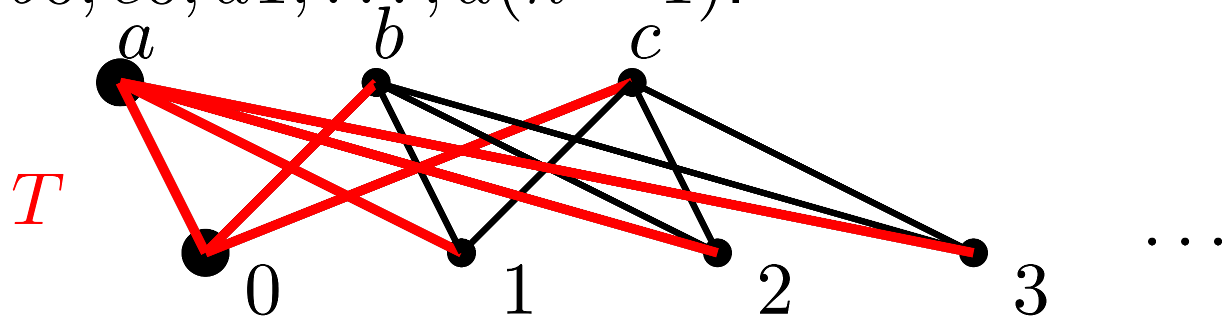

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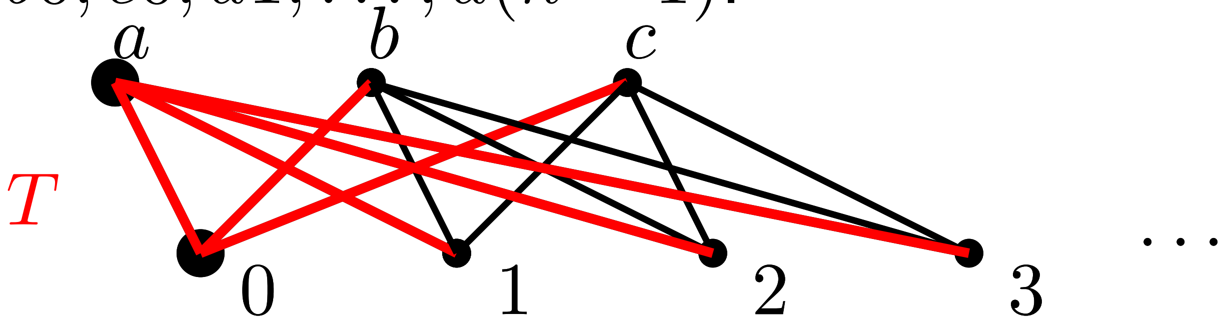

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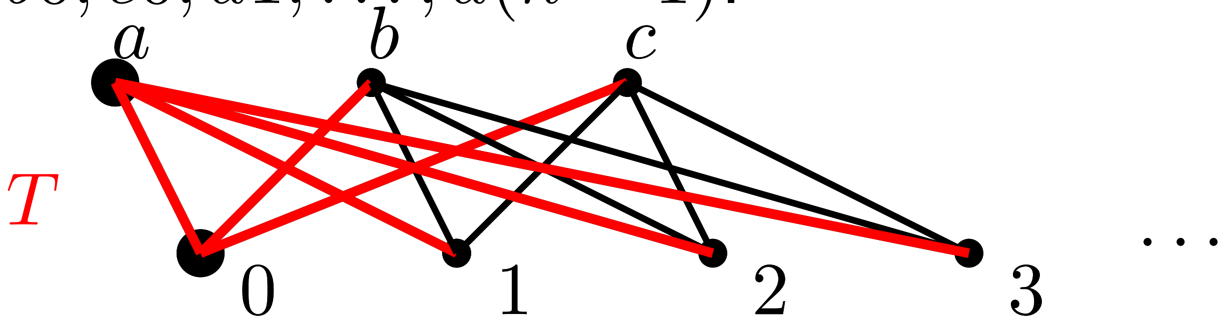

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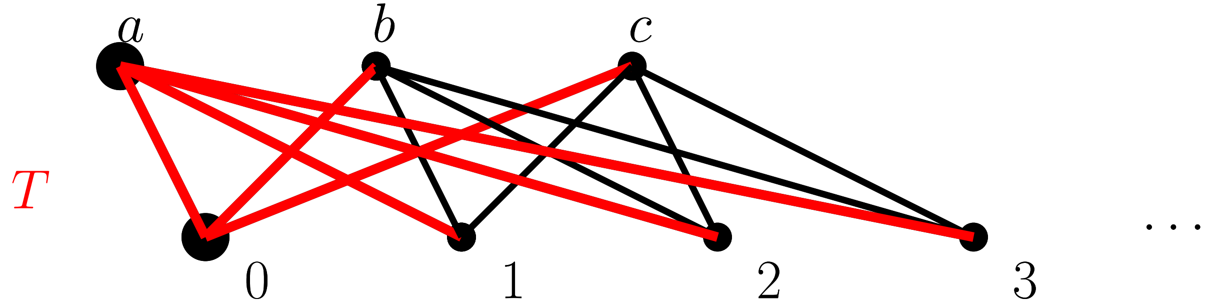

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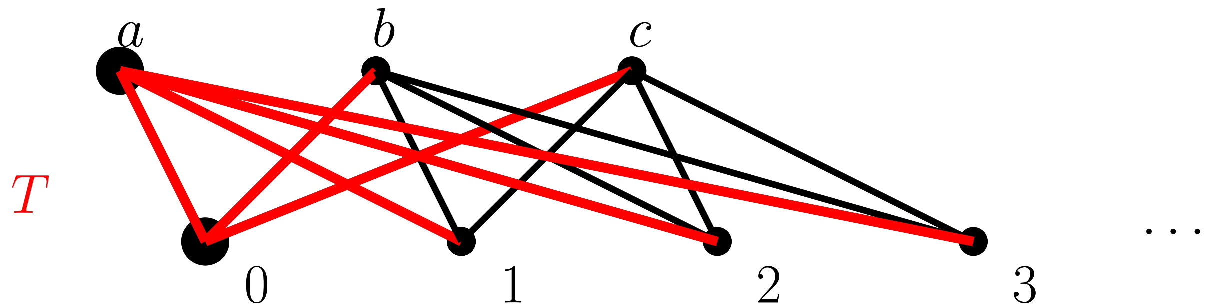

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Proof. $M + M^T = I_{n-1} + J_{n-1}$, and thus,
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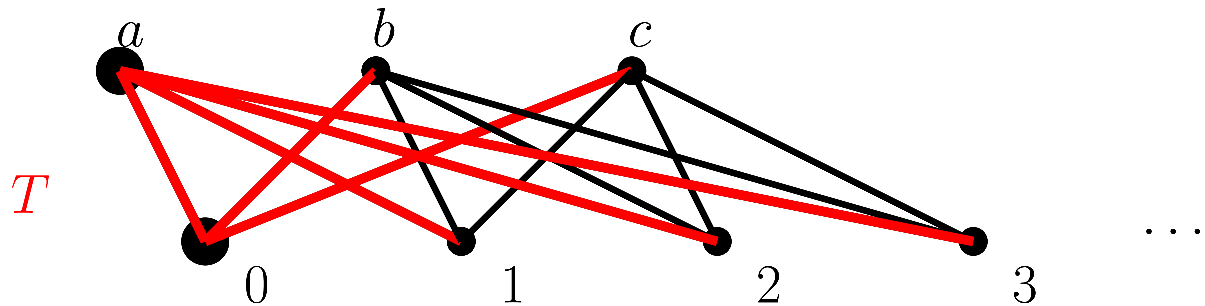

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Corollary 7. *If $K_{3,n}$ admits a $\mathbb{Z}_2$-embedding on a surface S then $K_{3,n}$ embeds on S.*

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Does $eg(G) = eg_0(G)$ where $eg(G)$ and $eg_0(G)$ is the Euler genus and Euler $\mathbb{Z}_2$-genus, respectively?(Conjecture by Schaefer and Štefankovič)