A Product Inequality for Extreme Distances

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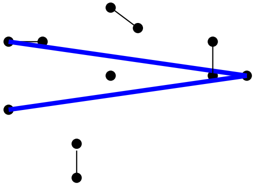
A Product Inequality

Let $p_1, \ldots, p_n$ be n distinct points in the plane, and assume that the minimum inter-point distance occurs $s_{\min}$ times, while the maximum inter-point distance occurs $s_{\max}$ times.



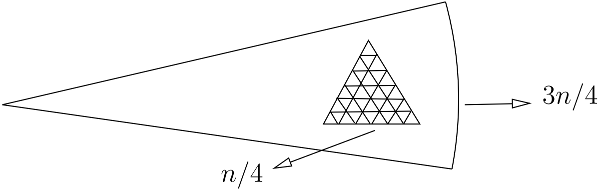
Erdős and Pach (1990) asked for a proof or disproof of the following product inequality:

$$s_{\min} s_{\max} \leq \frac{9}{8} n^2 + o(n^2).$$

Here it is shown that $s_{\min} s_{\max} \leq \frac{9}{8} n^2 + O(n)$.

A Product Inequality

The authors also remarked that this inequality, if true, essentially cannot be improved; and this would follow from a construction of E. Makai Jr. (not discussed in their paper).



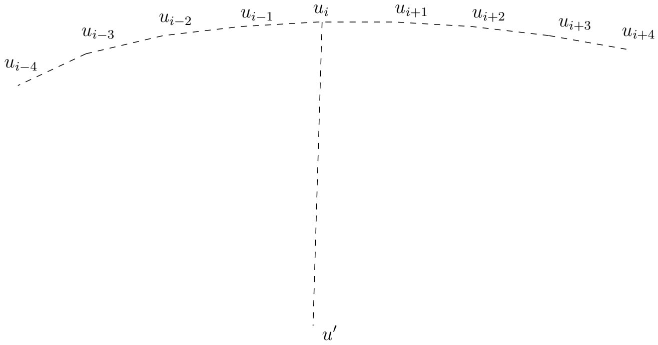
An n-element point set with $\frac{3}{4}n$ points on the convex hull and $\frac{1}{4}n$ interior points. $\frac{3}{4}n-1$ boundary points are evenly distributed on a circular arc centered at the leftmost point. $s_{\min} = \frac{3}{4}n + \frac{3}{4}n - O(\sqrt{n}) = \frac{3}{2}n - O(\sqrt{n})$, and $s_{\max} = \frac{3}{4}n$ (provided that the circular arc subtends an angle of 60°), and so $s_{\min}s_{\max} = \frac{9}{8}n^2 - O(n\sqrt{n})$. The $m = \frac{1}{4}n$ interior points make a section of a unit triangular lattice with $\lfloor 3m - \sqrt{12m-3} \rfloor$ unit distances, where the minimum inter-point distance is equal to 1.

Preliminaries

- Let $S = \{p_1, \ldots, p_n\}$ be a set of n distinct points in the plane. Given two points p and q, let $\ell(p, q)$ denote the line determined by p and q. Let δ and Δ denote the minimum and maximum pairwise distance of S, respectively. We may assume that $\delta = 1$; a standard packing argument yields $\Delta = \Omega(\sqrt{n})$. Let $G := G_\delta$ and G_Δ denote the respective graphs. It is well-known that $|E(G_\delta)| \leq 3n$ and $|E(G_\Delta)| \leq n$.
- For any point $u \in S$, let $\deg(u)$ denote its degree in G; it is well known that $\deg(u) \leq 6$ for any $u \in S$. For any point $u \in S$, let $\Gamma(u) = \{v \in S \ : \ uv \in E(G_\delta\}$; i.e., $\Gamma(u)$ is the set of vertices adjacent to u in G. For a point u, let $x(u)$ and $y(u)$ denote its x- and y-coordinates respectively.
- For a point set S, $\mathrm{conv}(S)$ denotes the convex hull of S, while $\partial\mathrm{conv}(S)$ denotes the boundary of $\mathrm{conv}(S)$.
- For a vertex $u \in H$, let u^- and u^+ denote the vertices that precede and succeed u, respectively, in clockwise order.

Proof setup

Let $H \subseteq S$ denote the set of (extreme) vertices of $\mathrm{conv}(S)$; labeled in clockwise order. We say that a vertex $u_i \in H$ has a *flat neighborhood* if the interior angles of the seven vertices u_{i-3}, u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}, and u_{i+3} all belong to the interval $(179°, 180°)$. Observe that the number of vertices of $\mathrm{conv}(S)$ that are not flat is $O(1)$.



The flat neighborhood of u_i and a diameter pair (u_i, u_i').

Proof setup

Let $F \subseteq H$ denote the set of (extreme) vertices of $\mathrm{conv}(S)$ that have flat neighborhoods. Let $D \subseteq H$ denote the set of (extreme) vertices of $\mathrm{conv}(S)$ that are endpoints of some diameter pair. Put $|D| = d$, $f = |F|$, and $h = |H|$; as such, $d \leq h$ and $f \leq h$.
The set of points S can be partitioned into three parts as $S = H \cup H' \cup I$, where

- H is the set of extreme vertices of $\mathrm{conv}(S)$; an element of H can be in any of the following sets $D \cap F$, $D \setminus F$, $F \setminus D$, or in neither of the two.
- H' is the set of points on $\partial\mathrm{conv}(S)$ that are not in H (the interior angle of each vertex in H' is $180°$).
- I is the set of interior vertices, i.e., those that are not on $\partial\mathrm{conv}(S)$.

Proof setup

As mentioned earlier, we have $s_{\max} \leq d \leq h$. Indeed, the endpoints of any diameter pair must be extreme points on the boundary of $\mathrm{conv}(S)$.

If $d \leq n/2$, then $s_{\max} \leq d \leq n/2$ and consequently, $s_{\min} s_{\max} \leq \frac{3}{2}n\frac{1}{2}n < n^2$, as required (with room to spare).

$\rightarrow$ We therefore subsequently assume that $h \geq d \geq n/2$.
Recall that $\delta = 1$; and $G := G_\delta$, and G_Δ is the *diameter* graph.

Lemma

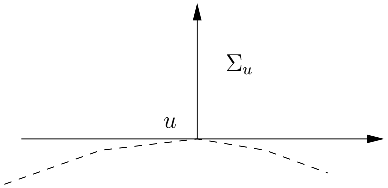
If $h \geq n/2$, then $\Delta \geq \frac{n}{2\pi}$; in particular $\Delta = \Omega(n)$.

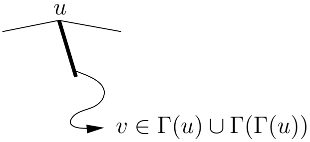
Proof.

Let $p = \mathrm{per}(\mathrm{conv}(S))$ Since $\delta = 1$ and $h \geq n/2$, we have $p \geq n/2$. By a standard isoperimetric inequality, $p \leq \pi\Delta$. Putting the two inequalities together yields $\Delta \geq \frac{n}{2\pi}$, as required. □

Proof setup: a rotating coordinate system used when charging u

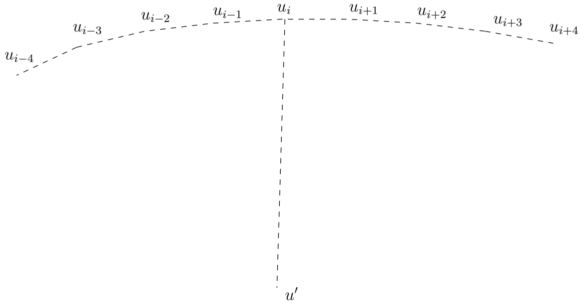
For any extreme vertex $u \in H$, let Σ_u be an orthogonal coordinate system whose origin is u, and where the x-axis is a supporting line of $\mathrm{conv}(S)$ incident to u, and S lies in the closed halfplane below the x-axis. If $uu^+ \in G$ and there exists $v \in I$ s.t. $vu, vu^+ \in G$, the x-axis of Σ_u will be chosen as the direction of next side (clockwise), $\overrightarrow{uu^+}$; otherwise, the x-axis of Σ_u will be chosen so that $S \setminus \{u\}$ lies strictly below this line.





Proof setup

Assume that each vertex $u_i \in D \cap F$ of degree 3 in G is charged to some interior vertex $v \in \Gamma(u_i) \cup \Gamma(\Gamma(u_i))$, of degree at most 5; so that the final charge of each interior vertex is at most 6; with each vertex receiving a charge at most 2.



Lemma

Let $u_i \in D \cap F$ be charged to some $v \in \Gamma(u_i) \cup \Gamma(\Gamma(u_i))$, where v is not necessarily unique. Then no vertex in $H \setminus \{u_{i-3}, u_{i-2}, u_{i-1}, u_{i+1}, u_{i+2}, u_{i+3}\}$ can send any charge to v.

An upper bound on $s_{\min}$ (from the same assumption)

Lemma

$s_{\min} \leq 3n - 2d + O(1)$.

Proof. Assume that that each element of I carries an initial charge equal to its degree in G (at most 6). Each element of H has degree at most 3; if $\deg(u) = 4$, then the interior angle at u equals $180°$, and so u is not an extreme vertex of $\mathrm{conv}(S)$. In particular, each element of $D \cap F$ has degree at most 3.

Observe that $|F \cap D| \geq |D| - O(1)$, since there are only $O(1)$ elements of D that do not have flat neighborhoods. Assuming the charging procedure complete, we have

$$\begin{aligned} 2s_{\min} &= \sum_{p \in S} \deg(p) \leq 3|H \setminus F \cap D| + 2|F \cap D| + 6|S \setminus H| \\ &= 3h - 3|F \cap D| + 2|F \cap D| + 6n - 6h \\ &= 6n - 3h - |F \cap D| \leq 6n - 3d - d + O(1) \\ &= 6n - 4d + O(1), \end{aligned}$$

as required. □

The resulting product inequality

Using the inequalities on $s_{\min}$ and $s_{\max}$:
$s_{\min} \leq 3n - 2d + O(1)$ and $s_{\max} \leq d$, we obtain

$$s_{\min} s_{\max} \leq (3n - 2d + O(1))\, d \leq \frac{9}{8} n^2 + O(n),$$

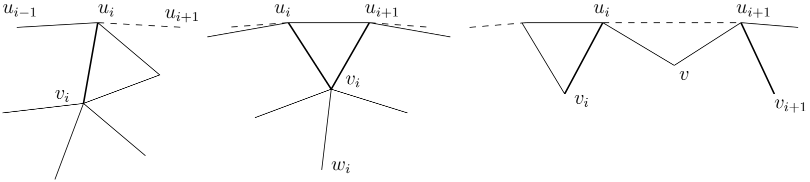
as required.

Indeed, setting $x = d/n$ yields the quadratic function $f(x) = x(3 - 2x)$, which attains its maximum value $\frac{9}{8}$ for $x = \frac{3}{4}$.

Thus $(3n - 2d)d \leq \frac{9}{8} n^2$, and we also have
$O(1)d = O(d) = O(n)$;
adding these two inequalities yields the one claimed above.

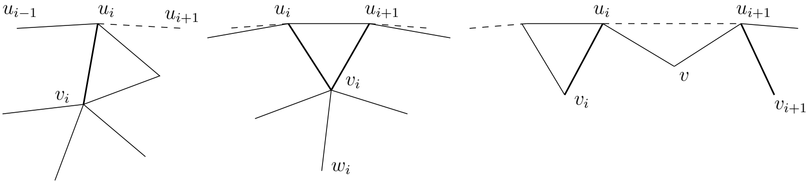
Charging scheme

Let $u_1, \ldots, u_h$ (where $u_{h+1} = u_1$) be the extreme vertices of $\mathrm{conv}(S)$ in clockwise order; they are processed one by one in this order (pairs of adjacent vertices of H corresponding to edges of G are processed at the same time); equivalently, S is rotated counterclockwise at each step so as the current vertex processed is the highest in the current step.



Charging rules

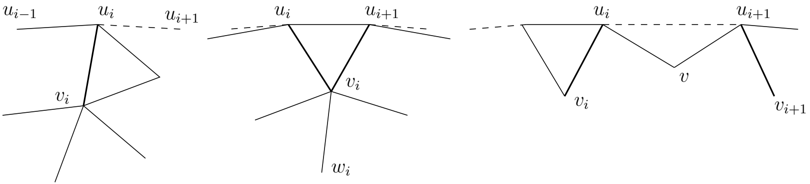
When handling the current vertex u_i (of deg. 3), or two consecutive vertices u_i, u_{i+1} that belong to a unit equilateral triangle, we use the coordinate system Σ_{u_i}. We distinguish several cases, depending on whether (i) the *middle* edge of unit length, say, $u_i v_i$, connects u_i with an interior vertex of degree 6 or less; and (ii) v_i is connected to one or two vertices on $\partial \mathrm{conv}(S)$.



Charging rules

The following charging rules are observed.

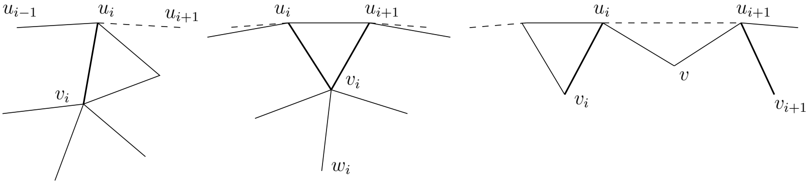
1. Only *middle* edges are charged (each to one or more interior vertices).
2. Charging amounts can be $1/2$ or 1.
3. Handling u_i (distribution of the unit charge on the middle edge incident to u_i) can only make charges to points at distance at most 2 in G; i.e., it can only affect vertices in $\Gamma(u_i) \cup \Gamma(\Gamma(u_i))$.



Middle edges are drawn in bold.

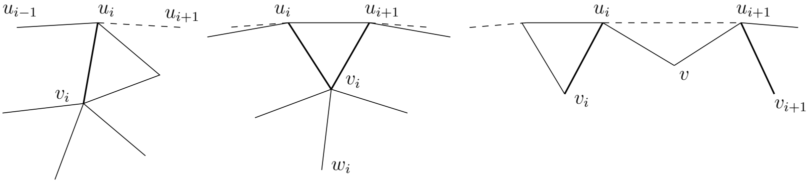
Charging rules

(a) If $u_i v_i$ is the unique unit edge incident to v_i connecting v_i with an extreme vertex, S is rotated counterclockwise, so that u_i is the highest vertex in S; see Fig. (left); the angle of rotation is set (arbitrarily) so this condition holds.



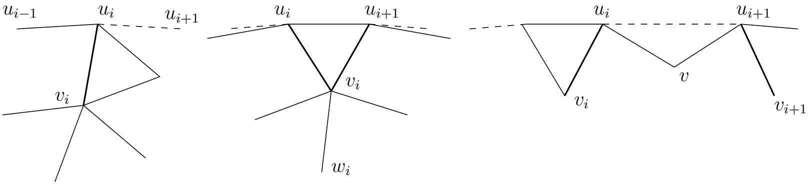
Charging rules

(b) If u_iv_i and $u_{i+1}v_i$ are unit edge incident to v_i connecting v_i with two adjacent extreme vertices u_i and u_{i+1} (i.e., $u_iu_{i+1} \in G$), S is rotated counterclockwise, so that u_iu_{i+1} is horizontal and S is contained in the closed halfplane below u_iu_{i+1}; see Fig. (middle).



Charging rules

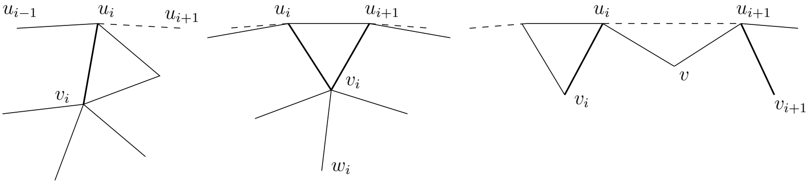
(c) If $u_i v$ and $u_{i+1}v$ are unit edge incident to v connecting v with two non-adjacent extreme vertices u_i and u_{i+1} (i.e., $|u_i u_{i+1}| > 1$), then $u_i v$ and $u_{i+1}v$ are *not* middle edges, and so we are in the situation described in (a) or (b); see Fig. (right), where middle edges $u_i v_i$ and $u_{i+1}v_{i+1}$ will be the ones charged to interior vertices.



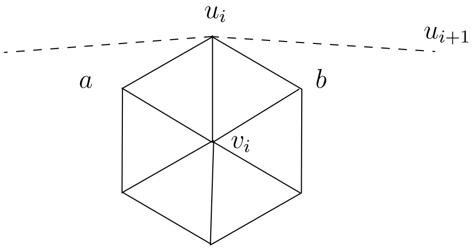
Properties

The following properties can be proven (as part of the charging scheme analysis).

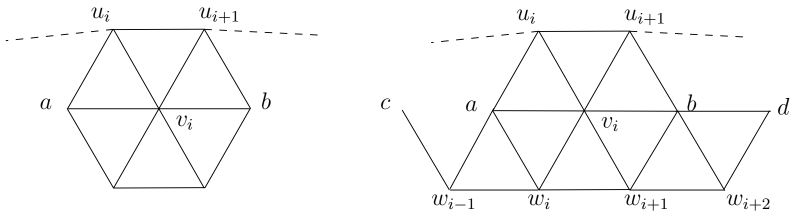
1. A vertex of degree 5 receives at most $1/2$ charge from the left, and at most $1/2$ charge from the right; or receives at most one unit of charge otherwise.
2. A vertex of degree at most 4 receives at most one unit of charge from the left, and at most one unit of charge from the right.
3. Write $u = u_i$. Consider the coordinate system Σ_u, and the rectangle $R_u = [x(u) - 7/4, x(u) + 7/4] \times [y(u) - 2, y(u)]$. By the charging scheme, u can only send charges to interior vertices contained in R_u.



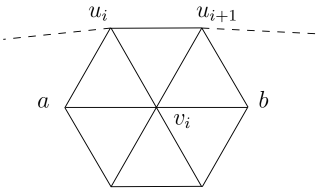
Case 1: $\deg(v_i) = 6$, and $u_i v_i$ is the unique unit edge incident to v_i connecting v_i with an extreme vertex. Let $a, b \in \Gamma(u_i) \cap \Gamma(v_i)$ be the other two common neighbors of u_i and v_i on the left and right, respectively. Note that $\deg(a) \leq 5$, and similarly, $\deg(b) \leq 5$; indeed, if $\deg(a) = 6$ (or $\deg(b) = 6$), one element in $\Gamma(a)$ (resp., $\Gamma(b)$) would lie strictly above u_i, a contradiction. Distribute the unit charge on edge $u_i v_i$ into two equal parts: $1/2$ to the left interior vertex a and $1/2$ to the right interior vertex b. Observe that $a, b \in R_{u_i}$. It will subsequently shown that the charge received by a (or b) from other nearby vertices on $\partial\text{conv}(S)$ is at most $1/2$.

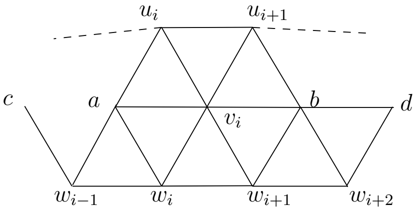


Case 2: $\deg(v_i) = 6$, where $u_i v_i$ and $u_{i+1} v_i$ are unit edge incident to v_i connecting v_i with two adjacent extreme vertices u_i and u_{i+1}. Note that $\deg(a) \leq 5$ and $\deg(b) \leq 5$; indeed, if say, $\deg(a) = 6$ (or $\deg(b) = 6$), the interior angle at u_i (resp., at u_{i+1}) would be $180°$, a contradiction, since we have assumed that $u_i, u_{i+1} \in D$. We further identify other vertices of low degree that will be charged. Let $w_i, w_{i+1} \in \Gamma(v_i)$ be the two neighbors of v_i below it. Our charging scheme is symmetric: we distribute the unit charge of edge $u_{i+1} v_i$ to b and some other interior vertex (the distribution of the unit charge of edge $u_i v_i$ is analogous, involving a and some other interior vertex).

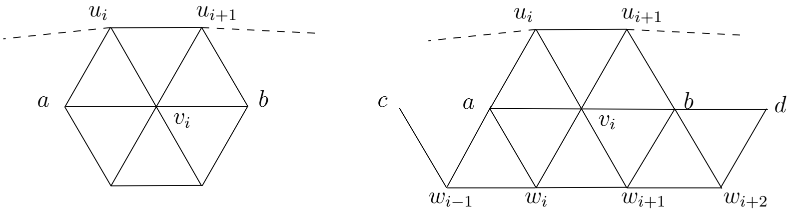


Case 2: $\deg(v_i) = 6$, where $u_i v_i$ and $u_{i+1} v_i$ are unit edge incident to v_i connecting v_i with two adjacent extreme vertices u_i and u_{i+1}. If $\deg(w_{i+1}) \le 5$, distribute the unit charge on edge $u_{i+1} v_i$ into two equal parts: $1/2$ to interior vertex b and $1/2$ to the interior vertex w_{i+1}. We subsequently assume that $\deg(w_{i+1}) = 6$. Let $w_{i+2} \in \Gamma(b) \cap \Gamma(w_{i+1})$ be the interior vertex on the line $\ell(w_i, w_{i+1})$ to the right. If $\deg(w_{i+2}) \le 5$, distribute the unit charge on edge $u_{i+1} v_i$ into two equal parts: $1/2$ to interior vertex b and $1/2$ to the interior vertex w_{i+2}. We subsequently assume that $\deg(w_{i+2}) = 6$, ...

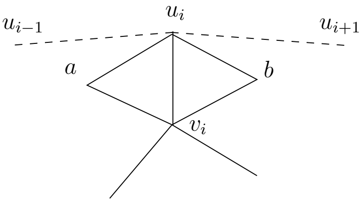


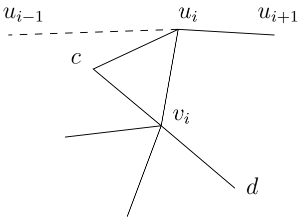


Case 2: $\deg(v_i) = 6$, where u_iv_i and $u_{i+1}v_i$ are unit edge incident to v_i connecting v_i with two adjacent extreme vertices u_i and u_{i+1}. We subsequently assume that $\deg(w_{i+2}) = 6$. Let $d \in \Gamma(b) \cap \Gamma(w_{i+2})$ be the interior vertex on the line $\ell(v_i, b)$ to the right. Observe that $\deg(d) \leq 4$: since each element of $\Gamma(d) \setminus \{b, w_{i+2}\}$ must lie strictly below the line $\ell(w_{i+2}, d)$, there are at most two such vertices. In this last case, distribute the unit charge on edge $u_{i+1}v_i$ into two equal parts: $1/2$ to the interior vertex b and $1/2$ to the interior vertex d. Observe that $b, d, w_{i+1}, w_{i+2} \in R_{u_{i+1}}$, and similarly that $a, c, w_i, w_{i-1} \in R_{u_i}$.

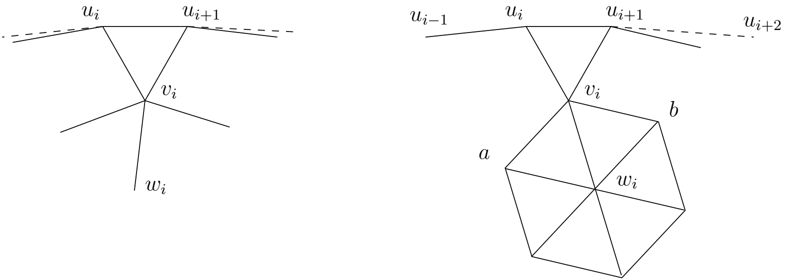


Case 3: $\deg(v_i) \leq 5$, and $u_i v_i$ is the unique unit edge incident to v_i connecting v_i with an extreme vertex. If $\deg(v_i) \leq 4$, v_i receives a unit charge. If $\deg(v_i) = 5$, let a and b be the two neighbors of v_i left and right of u_i, respectively. Let $\mathrm{high}(a, b)$ denote the element of $\{a, b\}$ which is the highest (i.e., closest to the x-axis of Σ_{u_i}). Observe that $\mathrm{high}(a, b)$ has degree at most 5; since otherwise, the y-coordinate of one of its neighbors (w.r.t. this coordinate system) would be non-negative, a contradiction. Further observe that $\mathrm{high}(a, b)u_i$ is an edge in G; since otherwise, u_i would not have degree 3 or its interior angle would be $180°$, either of which is a a contradiction. Distribute the unit charge on edge $u_i v_i$ into two equal parts: $1/2$ unit to v_i and $1/2$ unit to $\mathrm{high}(a, b)$.



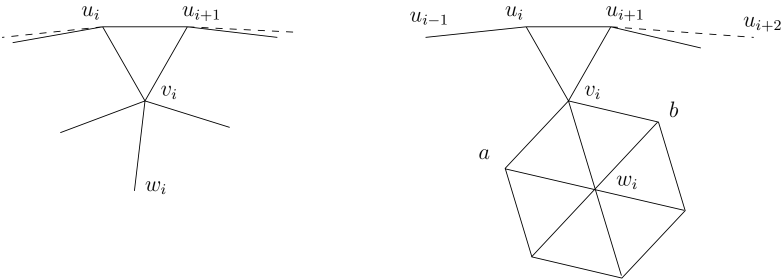


Case 4: $\deg(v_i) \leq 5$, where $u_i v_i$ and $u_{i+1} v_i$ are unit edge incident to v_i connecting v_i with two adjacent extreme vertices u_i and u_{i+1}.
If $\deg(v_i) \leq 4$, charge $u_i v_i$ and $u_{i+1} v_i$ to v_i; note that no other charge will be directed to this vertex. Assume now that $\deg(v_i) = 5$ and let w_i denote the vertex in $\Gamma(v_i)$ below v_i that is farthest from the edge $u_i u_{i+1}$.
If $\deg(w_i) \leq 5$, distribute the two units of charge for edges $u_i v_i$ and $u_{i+1} v_i$ into two equal parts: one unit to v_i and one unit to w_i.



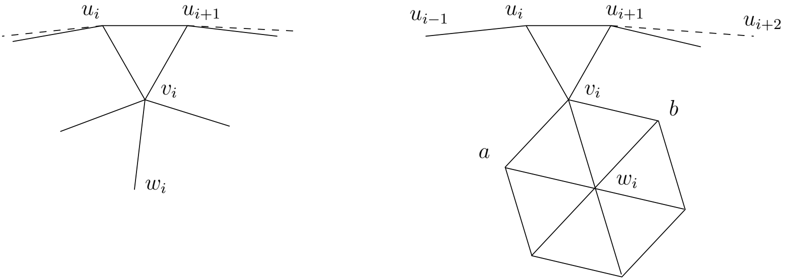
Left: $\deg(w_i) = 5$. Right: $\deg(w_i) = 6$.

Case 4: $\deg(v_i) \leq 5$, where $u_i v_i$ and $u_{i+1} v_i$ are unit edge incident to v_i connecting v_i with two adjacent extreme vertices u_i and u_{i+1}.
Assume now that $\deg(w_i) = 6$. We claim that $\deg(a) \leq 5$ and $\deg(b) \leq 5$. We may assume that $\angle a v_i u_i \geq 90° \geq \angle b v_i u_{i+1}$.
If $\deg(a) = 6$, let v_{i-1} be the next counterclockwise vertex after v_i in $\Gamma(a)$. Since the triangle $\Delta a v_{i-1} v_i$ is equilateral, this implies that $v_{i-1} v_i$ is yet another edge in G, which is in contradiction with the assumption that $\deg(v_i) = 5$.



Left: $\deg(w_i) = 5$. Right: $\deg(w_i) = 6$.

Case 4: $\deg(v_i) \leq 5$, where u_iv_i and $u_{i+1}v_i$ are unit edge incident to v_i connecting v_i with two adjacent extreme vertices u_i and u_{i+1}.
If $\deg(b) = 6$, then bu_{i+1} is an edge in G, thus $v_ib \parallel u_iu_{i+1}$ and so v_ib is horizontal. Let c be the next clockwise vertex after u_{i+1} in $\Gamma(b)$. Then $u_{i+1}c$ is also horizontal, thus $c \in \partial\text{conv}(S)$, which implies that the interior angle at u_{i+1} is $180°$, which is a contradiction (we have assumed that $u_i, u_{i+1} \in D$).
Distribute the two unit charges for edges u_iv_i and $u_{i+1}v_i$ as one unit to v_i, $1/2$ unit to a and $1/2$ unit to b.



Left: $\deg(w_i) = 5$. Right: $\deg(w_i) = 6$.

Further questions

1. What can be said about the maximum value of the product $s_{\min} s_{\max}$ in higher dimensions?
2. For the plane: Erdős and Pach (1990) also asked what is the best possible value of the constant c in the sum inequality below:

$$s_{\min} + s_{\max} \leq 3n - c\sqrt{n} + o(\sqrt{n}).$$

THE END