Topological Drawings of Complete Bipartite Graphs

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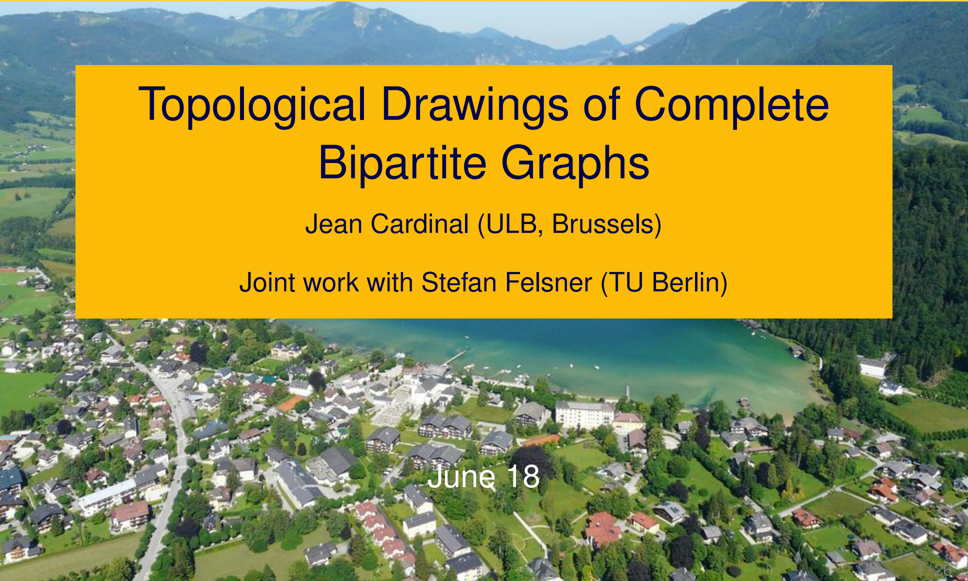
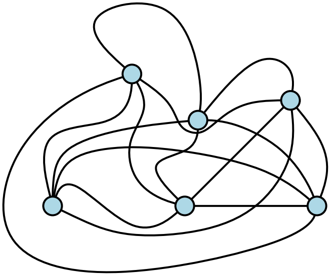
Joint work with Stefan Felsner (TU Berlin)



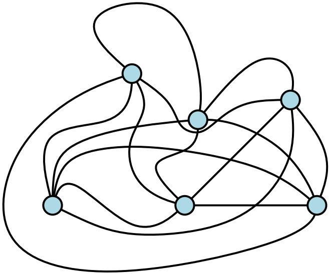
June 18

Topological Drawings of Graphs



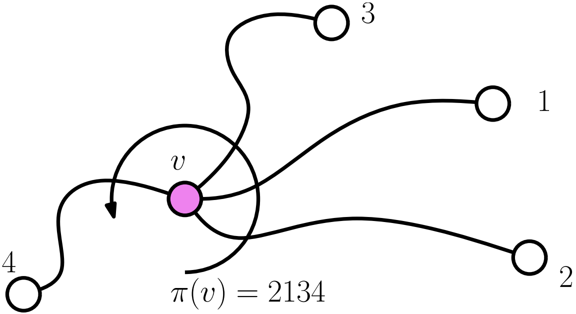
- vertices $\leftrightarrow$ points
- edges $\leftrightarrow$ (well-behaved) continuous curves

Simple Topological Drawings of Graphs



- vertices $\leftrightarrow$ points
- edges $\leftrightarrow$ (well-behaved) continuous curves **crossing pairwise at most once**

Simple Topological Drawings of **Complete** Graphs



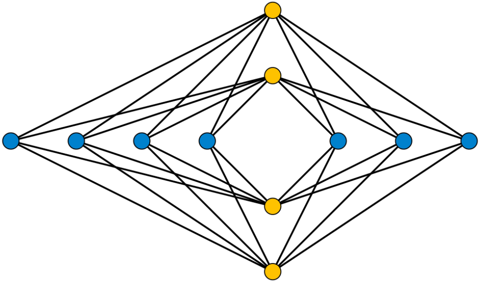
Rotation system ↔ crossing edges (Pach-Tóth 06)

Abstract Topological Graphs

- $G = (V, E, C)$, with $C \subseteq \binom{E}{2}$ **pairs of crossing edges**
- Simple realizability of complete AT-graphs **decidable in polynomial time** (Kyncl 11/15)

Topological Drawings of Complete **Bipartite** Graphs

- **Turán**'s brick factory problem
- **Zarankiewicz**'s conjecture

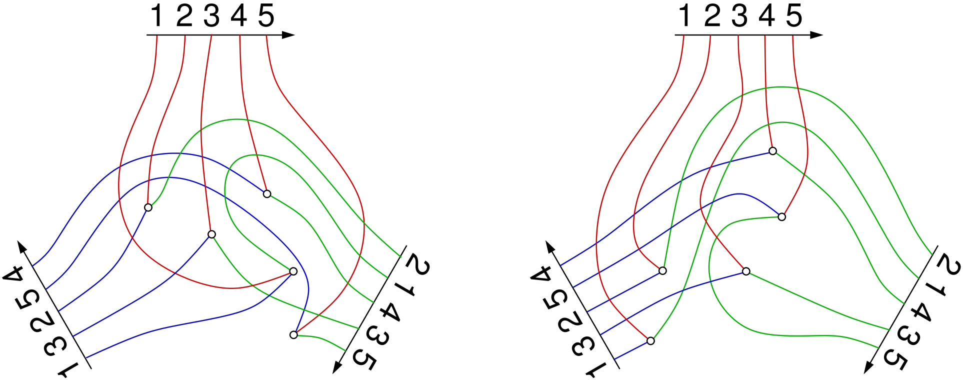


Outer Drawings of $K_{k,n}$

1. previous requirement of simple topological drawings **and**
2. the k vertices of one side of the bipartition lie on the outer boundary of the drawing.

Combinatorics of such drawings? Relevant combinatorial description and realizability checking?

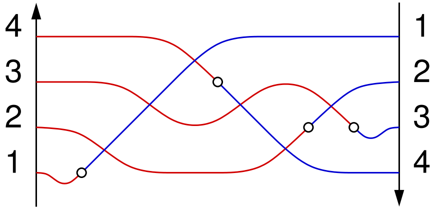
Examples



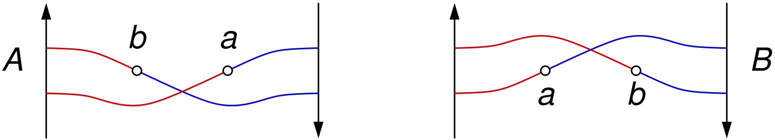
Outer drawings of $K_{3,5}$ with rotation system
$(12345, 21435, 13254)$

A first simple case

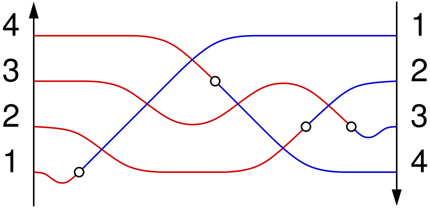
$k = 2$ and **uniform rotation system**



Encoding of $K_{2,2}$ subdrawings

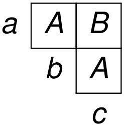


Example

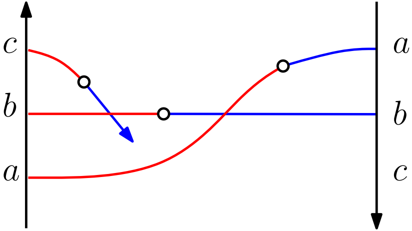


1	*B*	*B*	*B*
	2	*B*	*A*
		3	*A*
			4

Consistency constraints



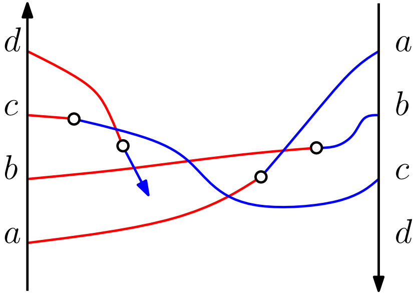
is not realizable



Triples are not enough

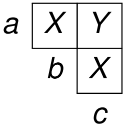
Only legal triples, but not realizable:

	b	c	d
a	*B*	*A*	*B*
b		*A*	*A*
c			*B*

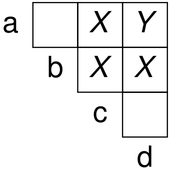


Drawings of $K_{2,4}$ yield **legal quadruples**

Triple and quadruple rules



$$\Rightarrow Y = X$$



$$\Rightarrow Y = X$$

Consistency for $k = 2$ and uniform rotation system

Theorem

Triple and quadruple consistency is sufficient for the existence of outer drawings of $K_{2,n}$ with uniform rotation system.

Structure

- Bijection with **separable permutations** = $\{2413, 3142\}$-avoiding permutations :

  triple rule $\Leftrightarrow$ permutation

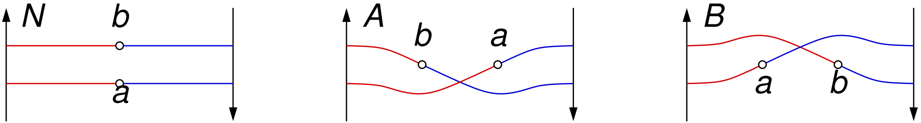
  quadruple rule $\Leftrightarrow$ pattern avoidance

Proof: consider the A, B matrices as matrices of inversions

Arbitrary k and arbitrary rotation system

- Generalization of the **triple and quadruple rules**
- Consider subdrawings of $K_{3,2}$ as well
- Sufficiency

Encoding of $K_{2,2}$ subdrawings



Triple rule

17 drawings of $K_{2,3}$ – **legal triples**

- 15 triples of the form

a	*X*	*Y*
	b	*Z*
		c

with $Y \in \{X, Z\}$

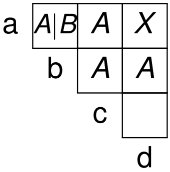
- 2 additional triples

a	*N*	*A*
	b	*B*
		c

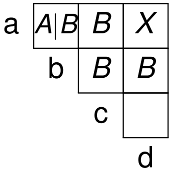
and

a	*A*	*B*
	b	*N*
		c

Quadruple rule

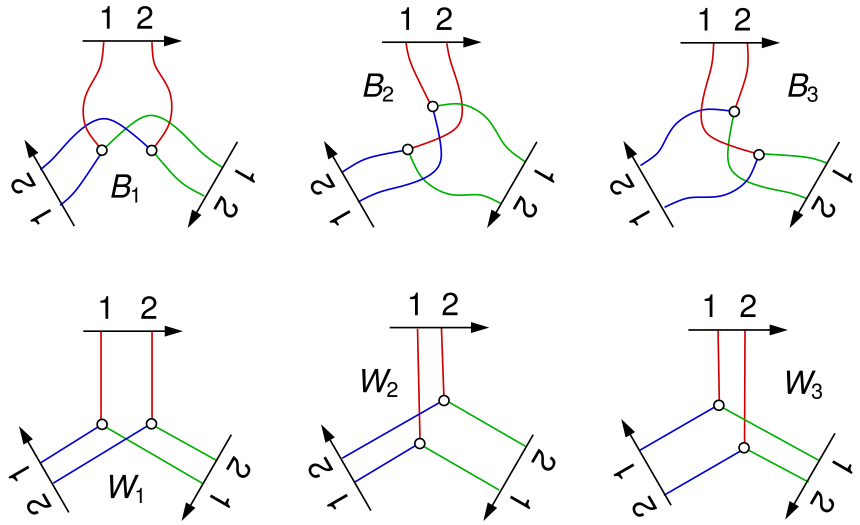


$\Rightarrow X = A$



$\Rightarrow X = B$

Drawings of $K_{3,2}$



Drawings of $K_{3,2}$: projections

	B_1	B_2	B_3	W_1	W_2	W_3
T_1	B	A	A	A	N	N
T_2	A	B	A	N	A	N
T_3	A	A	B	N	N	A

Consistency for arbitrary k

Theorem

Consistency on subdrawings of $K_{2,3}$ (triples), $K_{2,4}$ (quadruples), and $K_{3,2}$ is sufficient for the existence of outer drawings of $K_{k,n}$.

Corollary

Outer realizability of complete bipartite AT-graphs is in P

Proof steps

- $k = 2$ and arbitrary rotation system
- $k = 3$ and arbitrary rotation system : case analysis
- Generalize from $k = 3$ to arbitrary k

Other results

Rotation systems of **extendable** (aka **pseudolinear**)
outer drawings

$\leftrightarrow$

suballowable sequences (Asinowski 2008)

Thank you!

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