2.2 Partitioning Segments in $\mathbb{R}^2$

The main theorem is the following.

Theorem 2.3. *Let S be a set of n line segments in the plane with m intersections, in general position. Let $r \geq 1$ be a given parameter. Then there exists a partition of $\mathbb{R}^2$ into $O\left(r + \frac{mr^2}{n^2}\right)$ triangles such that the interior of any triangle in this partition intersects at most $\frac{n}{r}$ segments of S.*

Observe that this is asymptotically optimal. Given S, assume that there exists a partition of $\mathbb{R}^2$ with t triangles such that the interior of each triangle intersects at most $\frac{n}{r}$ segments of S. First, as there are n lines, we have $t \geq r$. Next, as the interior of each triangle can contain at most $\binom{\frac{n}{r}}{2} = O\left(\frac{n^2}{r^2}\right)$ intersection points, it must be that†

$$m \leq t \cdot O\left(\frac{n^2}{r^2}\right) \qquad \Longrightarrow \qquad t = \Omega\left(\frac{mr^2}{n^2}\right).$$

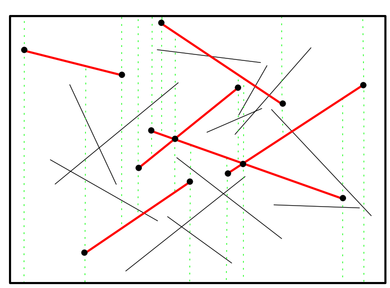
For the rest of this section, S will denote a set of n line segments in the plane in general position. For each $R \subseteq S$, let $I(R)$ denote the set of intersections between segments of R, and set $m_R = |I(R)|$.

We first briefly review a common way to partition space, the so-called *trapezoidal decompositions*.

Trapezoidal decompositions. Let S be a set of n line segments in the plane, and $\mathcal{U}$ a large-enough rectangle containing all the segments of S in its interior.

Then given any set $R \subseteq S$ of segments, partition $\mathcal{U}$ with respect to R as follows:

From each of the $2|R|$ endpoints of segments in R and each of the m_R intersection points between two segments of R, trace a vertical ray upwards and downwards until it hits another segment (or the bounding rectangle $\mathcal{U}$). The union of all these vertical segments, together with R, partitions $\mathcal{U}$ into a set of regions. Each such region is called a *trapezoidal region* (or a trapezoid), and the partition is called a *trapezoidal decomposition*.



†If S is in general position, then there can be only $O(t)$ intersects points on the boundary of the triangles of the partition.

Denote by $\Xi(R)$ this set of trapezoidal regions for a set R. The crucial fact that will be needed later is that each region $\Delta \in \Xi(R)$ in the trapezoidal decomposition is determined by a constant—2, 3 or 4—number of segments in R[†]. These are called the determining segments of Δ. The size of the trapezoidal decomposition of R, denoted by $|\Xi(R)|$, is the number of trapezoids in $\Xi(R)$.

The trapezoidal decomposition can be viewed as a planar graph—each of the $2|R|+m_R$ vertices consisting of the endpoints and intersections produce two additional points from the two rays, and the trapezoidal decomposition can be seen as a graph on these $3\,(2|R| + m_R)$ vertices. Thus we have

$$|\Xi(R)| \leq 3 \cdot 3\,(2|R| + m_R) = O\left(|R| + m_R\right).$$

Definition 2.1. *Given S, the set of trapezoids, over the trapezoidal decompositions of* all possible $R \subseteq S$*, are called the* canonical trapezoids *of S.*

For a canonical trapezoid Δ, let S_Δ denote the set of segments of S intersecting the interior of Δ. Then note the following fact.

Fact 2.4. A trapezoid Δ is present in the trapezoidal decomposition of R *if and only if* its determining segments are present in R, and R does not contain any of the segments of S_Δ.

For the rest of the proof, we only work with canonical trapezoids determined by 4 segments. The case for canonical trapezoids determined by 2 and 3 segments is similar.

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We return to our main theorem.

Theorem 2.3. *Let S be a set of n line segments in the plane with m intersections, in general position. Let $r \geq 1$ be a given parameter. Then there exists a partition of $\mathbb{R}^2$ into $O\left(r + \frac{mr^2}{n^2}\right)$ triangles such that the interior of any triangle in this partition intersects at most $\frac{n}{r}$ segments of S.*

Proof. We are given a set S of n line segments in the plane, with m pairs of pairwise intersecting segments.

First note that a slightly weaker bound—but within logarithmic factors—follows immediately from ϵ-nets.

Given S, consider the set system $(S, \mathcal{F})$ induced by intersection with triangles in the plane:

$$F \in \mathcal{F} \quad \text{if and only if} \quad \exists \text{ a triangle } \Delta \text{ such that } F = \{s \in S \colon s \cap \Delta \neq \emptyset\}.$$

[†]Recall that we assume S to be in general position.

We will compute a $\frac{1}{r}$-net R for $(S, \mathcal{F})$. However, as we will need a stronger property than just R being a $\frac{1}{r}$-net, we briefly recall the construction of R.

Let R be a random set constructed by uniformly choosing, for a large-enough constant C, each segment of S with probability

$$p = \frac{Cr \log r}{n}.$$

Following earlier ideas, it is not hard to show that then R is a $\frac{1}{r}$-net for $(S, \mathcal{F})$ with probability at least $\frac{9}{10}$.

> Geometrically, this means that *any* triangle Δ in the plane that intersects at least $\frac{1}{r} \cdot n$ segments of S must intersect a segment of R. Or put another way, any triangle Δ in the plane that *does not* intersect any segment of R intersects less than $\frac{1}{r} \cdot n$ segments of S.

One can triangulate the trapezoidal decomposition of R to get a triangulation $\mathcal{T}$ with asymptotically the same number of triangles. Now we claim that the interior of each triangle $\Delta \in \mathcal{T}$ must intersect less than $\frac{n}{r}$ segments of S. For contradiction, assume otherwise.

Shrink Δ slightly to get a triangle Δ' such that Δ' lies in the interior of Δ and any segment of S intersecting the interior of Δ intersects Δ'. But now Δ' does not intersect any segment of our sample R and intersects at least $\frac{n}{r}$ segments of S. But this contradicts the fact that R was a $\frac{1}{r}$-net.

It remains to bound the size of $\mathcal{T}$. As each point of S was picked into R independently with probability p, we have

$$\mathrm{E}\left[|R|\right] = np \quad = Cr \log r,$$

$$\mathrm{E}\left[m_R\right] = mp^2 = m\frac{C^2 r^2 \log^2 r}{n^2}.$$

By Markov's inequality, the probability that $|R| \geq 10np$ is at most $\frac{1}{10}$. Similarly the probability that m_R, the number of intersections between segments of R, is more than $10mp^2$ is also at most $\frac{1}{10}$. Thus, with probability at least $\frac{7}{10}$,

- R is an $\frac{1}{r}$-net for $(S, \mathcal{F})$, and
- the size of the trapezoidal decomposition of R is

$$|\Xi(R)| = O\left(|R| + |m_R|\right) = O\left(np + mp^2\right) = O\left(r \log r + \frac{mr^2 \log^2 r}{n^2}\right).$$

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We now remove the logarithmic factor. For a large-enough constant C, set

$$p = \frac{Cr}{n},$$

and pick each segment in S independently with probability p to get a random sample R.

Construct the trapezoidal decomposition $\Xi(R)$ of R. If all trapezoids $\Delta \in \Xi(R)$ intersect at most $\frac{n}{r}$ segments in S, we are done. Otherwise we will further partition each violating trapezoid—namely a trapezoid that intersects more than $\frac{n}{r}$ segments of S—based on two ideas.

First, the expected number of trapezoids in $\Xi(R)$ intersecting more than $\frac{n}{r}$ segments is small. In particular, we will show that, for any $t > 0$, the expected number of trapezoids intersecting at least $t \cdot \frac{n}{r}$ segments in S is an *exponentially* decreasing function of t.

Second, consider a $\Delta \in \Xi(R)$. Let

$$S_\Delta = \{s \in S \colon s \cap \Delta \neq \emptyset\}$$

$$n_\Delta = |S_\Delta|$$

$$m_\Delta = |\{p \in I(S) : p \in \Delta\}|.$$

Let $t > 0$ be such that $n_\Delta = t \cdot \frac{n}{r}$. Use the weaker bound, derived earlier, on S_Δ with parameter t, to get a partition inside Δ of

$$O\left(t \log t + \frac{m_\Delta t^2 \log^2 t}{n_\Delta^2}\right) = O\left(t \log t + \frac{n_\Delta^2 t^2 \log^2 t}{n_\Delta^2}\right) = O\left(t^2 \log^2 t\right)$$

trapezoids. By construction, each such new trapezoid inside Δ intersects at most $\frac{n_\Delta}{t} = \frac{n}{r}$ segments of S_Δ, and hence of S. Thus refining each Δ by adding new trapezoids, and taking the union of all these trapezoids for all $\Delta \in \Xi(R)$ gives the required partition on S with parameter r.

It remains to bound the overall expected size of this partition. Towards that we will need the two lemmas below.

Let $\Xi_{\leq k}$ be the set of canonical trapezoids defined by S that intersect at most k segments of S, i.e., those with $n_\Delta \leq k$.

Lemma 2.1.

$$|\Xi_{\leq k}| = O\left(nk^3 + mk^2\right).$$

Proof. Construct a random sample T by adding each segment of S to T with probability p_0. The expected total number of segments in T is np_0 and the expected number of intersections between the segments of T is mp_0^2.

The trick is to count the expected size of $\Xi(T)$ in two ways.

On one hand, it is

$$\mathrm{E}\left[|\Xi(T)|\right] = \mathrm{E}\left[O\left(|T| + m_T\right)\right] = \mathrm{E}\left[O\left(|T|\right)\right] + \mathrm{E}\left[O\left(m_T\right)\right] = O\left(np_0 + mp_0^2\right).$$

On the other hand, recalling that a trapezoid Δ appears in $\Xi(T)$ if and only if its four defining segments are picked in T, and none of the segments of S intersecting Δ are picked in T, we get that the probability of any fixed canonical trapezoid Δ appearing in $\Xi(T)$ is

$$p_0^4 \cdot (1-p_0)^{|\Delta \cap S|}.$$

Therefore the expected size of $\Xi(T)$ is

$$\mathrm{E}\left[|\Xi\left(T\right)|\right] = \sum_{\text{canonical } \Delta} p_0^4 \cdot (1-p_0)^{|\Delta \cap S|} \geq \sum_{\Delta \in \Xi_{\leq k}} p_0^4 \cdot (1-p_0)^{|\Delta \cap S|} \geq \sum_{\Delta \in \Xi_{\leq k}} p_0^4 (1-p_0)^k.$$

Putting the two bounds together,

$$|\Xi_{\leq k}| \cdot\ p_0^4(1-p_0)^k \ \leq\ \mathrm{E}\left[|\Xi\left(T\right)|\right] \ =\ O\left(np_0 + mp_0^2\right)$$

$$\Longrightarrow \qquad |\Xi_{\leq k}| = O\left(\frac{np_0 + mp_0^2}{p_0^4(1-p_0)^k}\right) = O\left(nk^3 + mk^2\right),$$

by setting $p_0 = \frac{1}{2k}$. □

Lemma 2.2. *For any $t > 0$, the expected number of trapezoids in $\Xi(R)$ which intersect at least $t \cdot \frac{n}{r}$ segments of S is*

$$O\left(\left(t^3 r + \frac{mr^2t^2}{n^2}\right) \cdot e^{-t}\right).$$

Proof. Consider first the expected number of trapezoids in $\Xi(R)$ which intersect $t\frac{n}{r}$ segments of S:

$$\mathrm{E}\left[\left|\Delta \in \Xi(R)\colon\ |\Delta \cap S| = \frac{tn}{r}\right|\right] = \left|\Delta \in \Xi(S)\colon\ |\Delta \cap S| = \frac{tn}{r}\right| \cdot p^4 \left(1-p\right)^{\frac{tn}{r}}$$

Using Lemma 2.1 and $p = \frac{Cr}{n}$, we get

$$= O\left(n\left(\frac{tn}{r}\right)^3 + m\left(\frac{tn}{r}\right)^2\right) \cdot \left(\frac{Cr}{n}\right)^4 e^{-p\frac{tn}{r}}$$

$$= O\left(\left(t^3 r + \frac{mr^2t^2}{n^2}\right) \cdot e^{-Ct}\right).$$

Observe that the above bound is decreasing exponentially in t, and therefore the required bound, which would follow by summing up over all trapezoids intersecting at least $\frac{tn}{r}$ segments in S, will be asymptotically the same:

$$\begin{aligned}
\sum_{\Delta:\, n_\Delta \geq tn/r} p^4 (1-p)^{n_\Delta} &= \sum_{i=0} \sum_{\frac{2^i tn}{r} \leq n_\Delta < \frac{2^{i+1} tn}{r}} p^4 (1-p)^{n_\Delta} \\
&\leq \sum_{i=0} \left(n \left(\frac{2^{i+1} tn}{r} \right)^3 + m \left(\frac{2^{i+1} tn}{r} \right)^2 \right) \cdot p^4 \cdot e^{-p \frac{2^i tn}{r}} \\
&\leq \sum_{i=0} \left(t^3 r 2^{3i+3} + \frac{m r^2 t^2 2^{2i+2}}{n^2} \right) e^{-C 2^i t} \\
&= t^3 r \left(\sum_{i=0} \left(2^{3i+3} \right) e^{-C 2^i t} \right) + \frac{m r^2 t^2}{n^2} \left(\sum_{i=0} \left(2^{2i+2} \right) e^{-C 2^i t} \right) \\
&= t^3 r \cdot O\left(e^{-Ct} \right) + \frac{m r^2 t^2}{n^2} \cdot O\left(e^{-Ct} \right).
\end{aligned}$$

This series is geometrically decreasing, so it is asymptotically equal to the required bound, for a large enough constant $C \geq 1$. □

Now we can complete the proof of the theorem. For each $\Delta \in \Xi(R)$, let t_Δ be such that

$$n_\Delta = t_\Delta \cdot \frac{n}{r}.$$

Using the weaker bound, refine trapezoid Δ by adding a $\frac{1}{t_\Delta}$-net R_Δ for all the $\frac{t_\Delta n}{r}$ segments of S intersected by Δ. The resulting expected total size of the trapezoidal partition is:

$$= |R| + \sum_{\Delta} \Pr\left[\Delta \in \Xi(R) \right] \cdot \left(\text{size of the decomposition of } \frac{1}{t_\Delta}\text{-net within } \Delta \right)$$

$$= |R| + \sum_{\Delta} \Pr[\Delta \in \Xi(R)] \cdot O\left(t_\Delta \log t_\Delta + \frac{m_\Delta t_\Delta^2 \log^2 t_\Delta}{n_\Delta^2} \right) \qquad \text{(using the weaker bound)}$$

$$\leq |R| + \sum_{\Delta} \Pr[\Delta \in \Xi(R)] \cdot O\left(t_\Delta^2 \log^2 t_\Delta \right) \qquad (\text{as } m_\Delta \leq n_\Delta^2)$$

$$= |R| + \sum_{j} \sum_{\substack{\Delta \text{ s.t.} \\ 2^j \leq t_\Delta < 2^{j+1}}} \Pr[\Delta \in \Xi(R)] \cdot O\left(t_\Delta^2 \log^2 t_\Delta \right)$$

$$\leq |R| + \sum_{j} \mathrm{E}\left[\text{ \# trapezoids } \Delta \text{ in } \Xi(R) \text{ with } 2^j \leq t_\Delta \right] \cdot O\left(2^{2(j+1)} \log^2 2^{j+1} \right)$$

$$\leq |R| + \sum_j O\left(\left(2^{3j} r + \frac{mr^2 2^{2j}}{n^2}\right) e^{-2^j}\right) \cdot O\left(2^{2(j+1)} \log^2 2^{j+1}\right) \qquad \text{(Lemma 2.2)}$$

$$= |R| + \left(r \sum_j O\left(2^{3j} e^{-2^j}\right) \cdot O\left(2^{2(j+1)} \log^2 2^{j+1}\right)\right) + \left(\frac{mr^2}{n^2} \sum_j O\left(2^{2j} e^{-2^j}\right) \cdot O\left(2^{2(j+1)} \log^2 2^{j+1}\right)\right)$$

$$= np + mp^2 + O(r) + O\left(\frac{mr^2}{n^2}\right) = O\left(r + \frac{mr^2}{n^2}\right) \qquad \text{(as the summands form a geometric series)}.$$

This finishes the proof of Theorem 2.3. □

Bibliography and discussion. The proof of the main theorem is from [1].

[1] M. de Berg and O. Schwarzkopf. Cuttings and applications. *Int. J. Comput. Geometry Appl.*, 5(4):343–355, 1995.

2.3 Application: Forbidden Subgraphs

The main result of this section is the following.

Theorem 2.5. *Let S be a set of n line segments in general position in the plane. If the intersection graph $G_I(S, E_I)$ of S does not contain $K_{t,t}$ as a subgraph, then $|E_I| = O(n)$, where the constant in the asymptotic notation depends only on t.*

A classical question in extremal graph theory is bounding the number of edges in graphs and hypergraphs not containing certain forbidden subgraphs. Consider the Zarankiewicz problem.

> Let $G = (V, E)$ be a graph on n vertices, and $t \geq 1$ be a given integer. What is the maximum size of E if G does not contain the subgraph $K_{t,t}$?

An early bound—and still the best known—is the following.

Theorem 2.6 (Kövári-Sós-Turán theorem)**.** *Let $G = (V, E)$ be a graph on n vertices, where G does not contain the subgraph $K_{t,t}$, for an integer $t \geq 1$. Then $|E| \leq n^{2-\frac{1}{t}}$.*

Proof. For any vertex $v \in V$, let $N_G(v)$ denote the neighbors of v in G.

We will double-count the following pairs.

$$T = \{(v, S) : v \in V \quad \text{and} \quad S \subseteq N_G(v), \ \ |S| = t\}.$$

On one hand, each S with $|S| = t$ can belong to at most $(t-1)$ tuples in T, as otherwise a $K_{t,t}$ would exist in G. So we have

$$|T| \leq \binom{n}{t} \cdot (t-1).$$

On the other hand, we can count $|T|$ exactly vertex by vertex:

$$|T| = \sum_{v \in V} \binom{|N_G(v)|}{t}.$$

Putting these bounds together[†] gives the required upper-bound on $\sum_v |N_G(v)| = 2|E|$. $\square$

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We consider the following geometric scenario. Let S be a set of n line segments in the plane. Assume that S is in general position, that is,

[†]Using Jensen's inequality.

- the intersection of every two segments of S is either empty, or is a point lying in the interior of both segments, and
- the three supporting lines of any three segments of S do not have a common intersection point.

Denote by $G_I(S, E_I)$ the intersection graph of S, namely

$$E_I = \Big\{ \{s, s'\} : s, s' \in S \quad \text{and} \quad s \cap s' \neq \emptyset \Big\}.$$

Note that $|E_I|$ is simply the number of intersections between the segments of S.

We return to the main result of this section, and prove it.

Theorem 2.5. *Let S be a set of n line segments in general position in the plane. If the intersection graph $G_I(S, E_I)$ of S does not contain $K_{t,t}$ as a subgraph, then $|E_I| = O(n)$, where the constant in the asymptotic notation depends only on t.*

Proof. Let $m = |E_I|$ be the number of intersections between the segments of S. Apply the segment partitioning bound, with the parameter r to be fixed later, to get a partition

$$\Xi(S) = \{\Delta_1, \dots, \Delta_t\}$$

of the plane into $t \leq C \cdot \left(r + \frac{mr^2}{n^2}\right)$ triangles, where C is a fixed constant. By increasing it if necessary, we can assume that $C \geq 2$.

Let $S_i \subseteq S$ be the set of segments that intersect the interior or boundary of Δ_i. Then for each Δ_i, we have that

- at most $\frac{n}{r}$ segments intersect its interior,
- there are at most 6 segments passing through the vertices of Δ_i, and
- at most two segments lie on any edge of Δ_i (by general position assumption) and thus there are at most 6 such segments.

Thus for each $i = 1, \dots, t$,

$$|S_i| \leq \frac{n}{r} + 12 \leq \frac{2n}{r},$$

assuming that $12 \leq \frac{n}{r}$ (our value of r, set later, will satisfy this).

Now consider an intersection point q lying in the interior or boundary of Δ_i. As q lies in the interior of both segments, it is not hard to see that both these segments must be present in S_i. By upper-bounding the intersections within each triangle of $\Xi(S)$ using the graph-theoretic bound of Theorem 2.6, we get

$$m \leq \sum_{\Delta_i \in \Xi(S)} \Big(\text{\# of intersections in the interior or boundary of } \Delta_i\Big)$$

$$\leq \sum_{\Delta_i \in \Xi(S)} |S_i|^{2-\frac{1}{t}} \leq \sum_{\Delta_i \in \Xi(S)} \left(\frac{2n}{r}\right)^{2-\frac{1}{t}} \leq 4C \cdot \left(r + \frac{mr^2}{n^2}\right) \cdot \left(\frac{n}{r}\right)^{2-\frac{1}{t}}.$$

Setting $r = \frac{n}{(8C)^t}$, we get

$$m \leq 4C \left(\frac{n}{(8C)^t} + \frac{m\left(\frac{n}{(8C)^t}\right)^2}{n^2}\right) \cdot \left((8C)^t\right)^{2-\frac{1}{t}} = 4C\left(\frac{n}{(8C)^t} \cdot \left((8C)^t\right)^{2-\frac{1}{t}} + \frac{m}{8C}\right)$$

$$\leq 4 \cdot 8^t C^{t+1} n + \frac{m}{2}$$

$$\implies m \leq (8C)^{t+1}\, n.$$

□

Bibliography and discussion. The proof is from [1].

[1] N. H. Mustafa and J. Pach. On the Zarankiewicz problem for intersection hypergraphs. *Journal of Combinatorial Theory, Series A*, 141:1–7, 2016.